Quantum Field Theory

Tizian Römer

Background Calculations

based on

Lecture “Theoretische Teilchenphysik I“ by Dieter Zeppenfeld,
Lectures “Quantum Field Theory” by Tobias Osborne on YouTube ➲,
Lecture Notes “Quantum Field Theory” by Gernot Eichmann ➲,
Lecture Notes “Quantum Field Theory” by David Tong ➲,
Lecture Notes “Modern Quantum Field Theory” by Einan Gardi ➲,
Lecture Notes “Quantum Field Theory” by Joachim Kopp ➲,
“The Quantum Theory of Fields” by Steven Weinberg,
“An Introduction to Quantum Field Theory” by Peskin, Schroeder.

This script is a supplement for the separated “Overview Script” and provides fully worked out calculations, which are omitted in the “Overview Script” for better overview.

Contact via E-Mail: tiroemer@yahoo.de

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2 Symmetries and Group Theory

2.1 Symmetry Transformations

2.1.1 Properties of the Expansion Coefficients of $\boldsymbol{U}$

The general expansion for the unitary operators is given by

$$U(\mathcal{T}(\theta)) = 1 + i\theta^a t^a + \frac{1}{2}\theta^a\theta^b t^{ab} + \mathcal{O}(\theta^3).$$

In this expansion the t's are operators/matrices. t^{ab} is symmetric in a, b, since any antisymmetric part of t^{ab} would vanish due to the symmetric $\theta^a\theta^b$, and t^a needs to be Hermitian (i.e. $t = t^\dagger$). This is easy to see, when we expand only up to the first order in θ, from which follows[1]

$$U^{-1} = 1 - i\theta^a t^a, \quad U^\dagger = 1 - i\theta^a t^{a\dagger}$$

and we already know that U is unitary, i.e. $U^{-1} = U^\dagger$.

2.1.2 Deriving the Relation between the generators

Consider the expansion of the equation

$$U(\mathcal{T}(\theta'))U(\mathcal{T}(\theta)) = U\left(\mathcal{T}(f(\theta', \theta))\right).$$

Up to second order, on the left-hand side we have

$$\begin{aligned} U(\mathcal{T}(\theta'))U(\mathcal{T}(\theta)) &= \left(1 + i\theta'^a t^a + \frac{1}{2}\theta'^a\theta'^b t^{ab}\right)\left(1 + i\theta^c t^c + \frac{1}{2}\theta^c\theta^d t^{cd}\right) \\ &= 1 + i\theta'^a t^a + i\theta^a t^a - \theta'^a\theta^b t^a t^b + \frac{1}{2}(\theta^a\theta^b + \theta'^a\theta'^b)t^{ab} \end{aligned}$$

and on the right-hand side

$$\begin{aligned} U\left(\mathcal{T}(f(\theta', \theta))\right) &= U\left(\mathcal{T}(\theta^a + \theta'^a + h^{abc}\theta^b\theta'^c)\right) \\ &= 1 + i(\theta^a + \theta'^a + h^{abc}\theta^b\theta'^c)t^a + \frac{1}{2}(\theta^a + \theta'^a + h^{acd}\theta^c\theta'^d)(\theta^b + \theta'^b + h^{bef}\theta^e\theta'^f)t^{ab} \\ &= 1 + i\theta^a t^a + i\theta'^a t^a + ih^{abc}\theta^b\theta'^c t^a + \frac{1}{2}(\theta^a\theta^b + \theta^a\theta'^b + \theta'^a\theta^b + \theta'^a\theta'^b)t^{ab}. \end{aligned}$$

If we equate those expressions and use that $t^{ab} = t^{ba}$ (see (>2.1.1)), we are left with

$$-\theta'^a\theta^b t^a t^b = ih^{abc}\theta^a\theta'^b t^c + \theta^a\theta'^b t^{ab},$$

which yields the relation

$$t^{ab} = -t^a t^b - ih^{abc}t^c.$$

2.1.3 The Lie Algebra

This can easily be seen, using (>2.1.2) and $t^{ab} = t^{ba}$:

$$[t^a, t^b] = t^a t^b - t^b t^a = (-t^{ab} - ih^{abc}t^c) - (-t^{ba} - ih^{bac}t^c) = i(h^{bac} - h^{abc})t^c =: if^{abc}t^c.$$

[1] The inverse is constructed such that $UU^{-1} = 1$:

$$(1 + i\theta^a t_a)(1 - i\theta^a t_a) = 1 + \mathcal{O}(\theta^2).$$

The adjoint is constructed simply by putting daggers everywhere and using $a^\dagger = a$ for $a \in \mathbb{R}$ as well as $i^\dagger = -i$.

2.1.4 The Jacobi Identity

In general, commutators obey the Jacobi identity

$$\big[A,[B,C]\big]+\big[B,[C,A]\big]+\big[C,[A,B]\big]=0.$$

Pluging in $A=t^a, B=t^b, C=t^c$ and using the Lie algebra, we find

$$\begin{aligned}0&=\left[t^a,[t^b,t^c]\right]+\left[t^b,[t^c,t^a]\right]+\left[t^c,[t^a,t^b]\right]=if^{bcd}[t^a,t^d]+if^{cad}[t^b,t^d]+if^{abd}[t^c,t^d]\\&=i^2 f^{bcd}f^{ade}t^e+i^2 f^{cad}f^{bde}t^e+i^2 f^{abd}f^{cde}t^e\end{aligned}$$

$$\Leftrightarrow\qquad f^{bcd}f^{dae}+f^{cad}f^{dbe}+f^{abd}f^{dce}=0.$$

Note that, by definition, $f^{abc}=-f^{bac}$. We used this identity in the last step for the second structure constant of each term (producing a global minus sign, which is irrelevant).

2.2 The SU(N) Group

2.2.1 Determinant of Exponential gives the Trace

Consider a Hermitian matrix A, which can be diagonalized by unitary matrices U like $A=U^\dagger DU$. Using the fact that

$$\det AB=\det BA,\qquad \operatorname{Tr}AB=\operatorname{Tr}BA,$$

we find

$$\begin{aligned}\det e^{iA}&=\det e^{iU^\dagger DU}=\det\sum_{n=0}^{\infty}\frac{\left(iU^\dagger DU\right)^n}{n!}=\det U^\dagger\sum_{n=0}^{\infty}\frac{(iD)^n}{n!}U=\det e^{iD}\\&=\det\begin{pmatrix}e^{i\lambda_1}&0&0\\0&\ddots&0\\0&0&e^{i\lambda_m}\end{pmatrix}=\prod_{i=1}^{m}e^{i\lambda_i}=e^{i\sum_{i=1}^m\lambda_i}=e^{i\operatorname{Tr}D}=e^{i\operatorname{Tr}U^\dagger DU}=e^{i\operatorname{Tr}A}.\end{aligned}$$

2.2.2 Explicit Formula for the Structure Constant

Using the normalization condition $\operatorname{Tr}t^at^b=T(R)\,\delta^{ab}$, where $T(R)$ is a number, we find

$$\operatorname{Tr}([t^a,t^b]t^c)=if^{abd}\operatorname{Tr}(t^dt^c)=if^{abd}\,T(R)\delta^{dc}=if^{abc}\,T(R)$$

$$\Leftrightarrow\qquad f^{abc}=-\frac{i}{T(R)}\operatorname{Tr}([t^a,t^b]t^c).$$

Already by its definition it is obvious that the structure constant is antisymmetric in the first two indices: $f^{abc}=-f^{bac}$. This equation implies, however, that the structure constants are even *totally* antisymmetric. Using the cyclicity of the trace and the general commutator identity $[t^a,t^bt^c]=t^b[t^a,t^c]+[t^a,t^b]t^c$, we find

$$\operatorname{Tr}([t^a,t^b]t^c)=\operatorname{Tr}([t^a,t^bt^c]-t^b[t^a,t^c])=-\operatorname{Tr}(t^b[t^a,t^c])=-\operatorname{Tr}([t^a,t^c]t^b),$$

where we used that the trace of a pure commutator always vanishes due to the cyclicity of the trace: $\operatorname{Tr}[A,B]=\operatorname{Tr}AB-\operatorname{Tr}BA=0$. In this way, swapping the order of to generators will always generate a minus sign.

2.2.3 Two Generators make a Casimir Invariant

We claimed that t^at^a is a Casimir invariant. Let's proof this:

$$[t^at^a,t^b]=t^a[t^a,t^b]+[t^a,t^b]t^a=if^{abc}t^at^c+if^{abc}t^ct^a=if^{abc}\{t^a,t^c\}=0.$$

This vanishes, since f^{abc} is antisymmetric in a,c, whereas $\{t^a,t^c\}$ is symmetric in a,c.

2.2.4 The Casimir Invariant in the Fundamental and Adjoint Representations

For the definitions of the fundamental and adjoint representations see the end of section 2.2.

Using the Fierz identity of the fundamental representation, we find in this representation

$$\delta_{ik}\, C_F \coloneqq \delta_{ik}\, C_2(\text{fund}) = t^a_{ij} t^a_{jk} = \frac{1}{2}\Big(\delta_{ik}\delta_{jj} - \frac{1}{N}\delta_{ij}\delta_{jk}\Big) = \frac{1}{2}\Big(N - \frac{1}{N}\Big)\,\delta_{ik} = \frac{1}{2}\Big(\frac{N^2-1}{N}\Big)\,\delta_{ik}$$

$$\Leftrightarrow \qquad C_F = \frac{N^2-1}{2N}.$$

Here we used $\delta_{jj} = N$; since we are in the fundamental representation, the matrices t^a have dimension N and thus the indices take on values $i, j, k = 1, \dots, N$.

To find the Casimir invariant in the adjoint representation, consider first the following computation:

$$\begin{aligned} t^a t^b t^a &= t^a(t^a t^b + [t^b, t^a]) = C_2 t^b - i f^{abc} t^a t^c = C_2 t^b - \frac{1}{2} i f^{abc}([t^a, t^c] + \{t^a, t^c\}) \\ &= C_2 t^b - \frac{1}{2} i f^{abc}(i f^{acd} t^d + 0) = C_2 t^b - \frac{1}{2}\big(t^a_{\text{adj}}\big)_{bc}\big(t^a_{\text{adj}}\big)_{cd} t^d = C_2 t^b - \frac{1}{2} C_A \delta^{bd} t^d \\ &= \Big(C_2 - \frac{1}{2} C_A\Big)\, t^b. \end{aligned}$$

Note, that this computation made no use of t^a beeing of some specific representation. Obviously, C_A is the Casimir invariant in the adjoint representation only, but still this equation holds for any representation of the t's when taking C_2 in the same representation. That is, it also holds in the fundamental representation. Evaluating $t^a t^b t^a$ in the fundamental representation explicitly, using the Fierz identity, we find

$$(t^a t^b t^a)_{il} = t^a_{ij} t^b_{jk} t^a_{kl} = \frac{1}{2}\Big(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\Big)\, t^b_{jk} = \frac{1}{2}\Big(\delta_{il} t^b_{jj} - \frac{1}{N} t^b_{il}\Big) = -\frac{1}{2N} t^b_{il},$$

where we used that t^a are traceless. Thereby, marking now explicitly the representations:

$$(t^a t^b t^a)^{\text{fund}} = \Big(C_F - \frac{1}{2} C_A\Big)\, t^b_{\text{fund}} = -\frac{1}{2N} t^b_{\text{fund}}$$

$$\Leftrightarrow \qquad C_F - \frac{1}{2} C_A = -\frac{1}{2N} \qquad \Leftrightarrow \qquad C_A = 2C_F + \frac{1}{N} = 2\frac{N^2-1}{2N} + \frac{1}{N} = N.$$

2.2.5 Proof of Relationship between Normalization and Casimir Invariant

From the definitions of the normalization $T(R)$ and the Casimir invariant $C_2(R)$,

$$\operatorname{Tr} t^a t^b = T(R)\,\delta^{ab}, \qquad t^a t^a = \mathbb{I}\, C_2(R)$$

follows

$$C_2(R) \cdot \dim R = C_2(R) \operatorname{Tr} \mathbb{I}_R = \operatorname{Tr}(C_2(R)\, \mathbb{I}_R) = \operatorname{Tr} t^a t^a = T(R)\,\delta^{aa} = T(R)\,(N^2 - 1),$$

where $\dim R$ is the dimension of the matrices of representation R and $\mathbb{I}_R$ is the unit matrix with that dimension $\dim R$.

2.2.6 Matrices of the Adjoint Representation Fulfil Lie Algebra

In (>2.1.4) we found the Jacobi identity (here multiplied by i^2)

$$i^2 f^{bcd} f^{dae} + i^2 f^{cad} f^{dbe} = -i^2 f^{abd} f^{dce}.$$

Let's commute the indices in the following way. Then, since in the adjoint representation, $(t^a)_{bc} = -i f^{abc}$, we can plug in the generators:

$$-i^2 f^{bcd} f^{ade} + i^2 f^{acd} f^{bde} = -i^2 f^{abd} f^{dce}$$

$$\Leftrightarrow \qquad -(t^b)_{cd}(t^a)_{de} + (t^a)_{cd}(t^b)_{de} = if^{abd}(t^d)_{ce}$$

$$\Leftrightarrow \qquad -(t^b t^a)_{ce} + (t^a t^b)_{ce} = if^{abd}(t^d)_{ce}$$

$$\Leftrightarrow \qquad [t^a, t^b]_{ce} = if^{abd}(t^d)_{ce}$$

$$\Leftrightarrow \qquad [t^a, t^b] = if^{abd} t^d.$$

Thus, the adjoint representation indeed fulfils the Lie algebra.

2.3 Lorentz Transformation

2.3.1 Deriving Properties of Lorentz Transformations

Any coordinate transformation obeying the equations above is linear,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu},$$

with arbitrary constants a^{μ} and constants $\Lambda^{\mu}{}_{\nu}$ satisfying the conditions

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma} \quad \Leftrightarrow \quad \Lambda^{\mu}{}_{\sigma}\Lambda^{\kappa}{}_{\tau}\eta^{\sigma\tau} = \eta^{\mu\kappa}, \quad \eta_{\mu\nu}\eta^{\nu\sigma} = \delta^{\sigma}_{\mu}$$

This equivalency can be derived by multiplying by $\Lambda^{\kappa}{}_{\tau}\eta^{\sigma\tau}$ and then by $\Lambda^{-1\,\rho}{}_{\epsilon}$ and finally by $\eta^{\mu\epsilon}$ on both sides. By the way, the inverse can be given as

$$\Lambda^{-1\,\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu} = \eta_{\nu\sigma}\eta^{\mu\rho}\Lambda^{\sigma}{}_{\rho}, \quad \text{since} \quad \Lambda_{\nu}{}^{\mu}\Lambda^{\nu}{}_{\kappa} = \left(\eta_{\nu\sigma}\eta^{\mu\rho}\Lambda^{\sigma}{}_{\rho}\right)\Lambda^{\nu}{}_{\kappa} = \eta^{\mu\rho}\eta_{\rho\kappa} = \delta^{\mu}{}_{\kappa}.$$

Taking the determinant yields

$$\det \Lambda^{\mu}{}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma} = \det\left(\Lambda^{T}\eta\Lambda\right) = \det^2 \Lambda \det \eta \overset{!}{=} \det \eta \quad \Leftrightarrow \quad \det \Lambda = 1.$$

2.3.2 Two successive Lorentz Transformations is a Lorentz Transformation Again

$\widetilde{\Lambda}^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma}$ also fulfills the condition $\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$:

$$\eta_{\mu\nu}\left(\widetilde{\Lambda}^{\mu}{}_{\kappa}\Lambda^{\kappa}{}_{\rho}\right)\left(\widetilde{\Lambda}^{\nu}{}_{\tau}\Lambda^{\tau}{}_{\sigma}\right) = \underbrace{\eta_{\mu\nu}\widetilde{\Lambda}^{\mu}{}_{\kappa}\widetilde{\Lambda}^{\nu}{}_{\tau}}_{=\eta_{\kappa\tau}}\Lambda^{\kappa}{}_{\rho}\Lambda^{\tau}{}_{\sigma} = \eta_{\rho\sigma}.$$

2.4 The Poincaré Algebra

2.4.1 Infinitesimal Lorentz Transformation

For an infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, \quad a^{\mu} = \epsilon^{\mu}$$

the Lorentz condition reads

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\left(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}\right) = \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + \mathcal{O}(\omega^2) \quad \Leftrightarrow \quad \omega_{\mu\nu} = -\omega_{\nu\mu}.$$

2.4.2 Lorentz Transformation of $\boldsymbol{P^{\mu}}$ and $\boldsymbol{J^{\mu\nu}}$

To first order in ω and ϵ, we have

$$U(\Lambda, a)\left(\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_{\mu}P^{\mu}\right)U^{-1}(\Lambda, a) = \frac{1}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu} - (\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_{\mu}P^{\mu}.$$

Equating the terms with $\omega_{\mu\nu}$ yields

$$U(\Lambda, a)\omega_{\sigma\rho}J^{\sigma\rho}U^{-1}(\Lambda, a) = \Lambda_{\mu}^{\sigma}\Lambda_{\nu}^{\rho}(J^{\mu\nu} + 2a^{\nu}P^{\mu})\omega_{\sigma\rho} = \Lambda_{\mu}^{\sigma}\Lambda_{\nu}^{\rho}(J^{\mu\nu} - a^{\mu}P^{\nu} + a^{\nu}P^{\mu})\omega_{\sigma\rho},$$

where the following identities where used:

$$\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu} = \Lambda_\mu{}^\sigma \omega_{\sigma\rho} \Lambda^{-1\,\rho}{}_\nu = \Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho \omega_{\sigma\rho},$$
$$\left(\Lambda\omega\Lambda^{-1} a\right)_\mu = \left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu} a^\nu = \Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho \omega_{\sigma\rho} a^\nu,$$
$$\Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho a^\nu P^\mu \omega_{\sigma\rho} = \Lambda_\nu{}^\rho \Lambda_\mu{}^\sigma a^\mu P^\nu \omega_{\rho\sigma} = -\Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho a^\mu P^\nu \omega_{\sigma\rho}.$$

Equating the terms with ϵ_μ read $U(\Lambda, a)\epsilon_\sigma P^\sigma U^{-1}(\Lambda, a) = \Lambda_\mu{}^\sigma \epsilon_\sigma P^\mu$, thus the two following equations hold:

$$U(\Lambda, a) J^{\sigma\rho} U^{-1}(\Lambda, a) = \Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho (J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu),$$
$$U(\Lambda, a) P^\sigma U^{-1}(\Lambda, a) = \Lambda_\mu{}^\sigma P^\mu.$$

2.4.3 Lie Algebra of the Poincaré Group (Commutator Relations of $\boldsymbol{P^\mu}$ and $\boldsymbol{J^{\mu\nu}}$)

Up to first order in ω and ϵ we get for the $J^{\sigma\rho}$-equation, using $(1+B)A(1-B) = A + [B, A] + \mathcal{O}(B^2)$,

$$\text{LHS:} \quad U(1+\omega, \epsilon) J^{\sigma\rho} U^{-1}(1+\omega, \epsilon) = U(1+\omega, \epsilon) J^{\sigma\rho} U(1-\omega, -(1-\omega)\epsilon)$$
$$= \left(1 + \left(\frac{1}{2} i\omega_{\mu\nu} J^{\mu\nu} - i\epsilon_\mu P^\mu\right)\right) J^{\sigma\rho} \left(1 - \left(\frac{1}{2} i\omega_{\kappa\tau} J^{\kappa\tau} - i\epsilon_\kappa P^\kappa\right)\right)$$
$$= J^{\sigma\rho} + \left[\left(\frac{1}{2} i\omega_{\mu\nu} J^{\mu\nu} - i\epsilon_\mu P^\mu\right), J^{\sigma\rho}\right],$$
$$\text{RHS:} \quad \left(\delta_\mu{}^\sigma + \omega_\mu{}^\sigma\right)\left(\delta_\nu{}^\rho + \omega_\nu{}^\rho\right)(J^{\mu\nu} - \epsilon^\mu P^\nu + \epsilon^\nu P^\mu) = J^{\sigma\rho} - \epsilon^\sigma P^\rho + \epsilon^\rho P^\sigma + \omega_\mu{}^\sigma J^{\mu\rho} + \omega_\nu{}^\rho J^{\sigma\nu}$$
$$\Leftrightarrow \quad i\left[\left(\frac{1}{2}\omega_{\mu\nu} J^{\mu\nu} - \epsilon_\mu P^\mu\right), J^{\sigma\rho}\right] = \omega_\mu{}^\sigma J^{\mu\rho} + \omega_\mu{}^\rho J^{\sigma\mu} - \epsilon^\sigma P^\rho + \epsilon^\rho P^\sigma.$$

Similarly, the P^μ-equation reads

$$i\left[\left(\frac{1}{2}\omega_{\mu\nu} J^{\mu\nu} - \epsilon_\mu P^\mu\right), P^\sigma\right] = \omega_\mu{}^\sigma P^\mu.$$

Again, by equating the coefficients of $\omega_{\mu\nu}$ and ϵ_μ on both sides, the following commutator relations follow (note that $\omega_{\mu\nu} A^{\mu\nu} = -\omega_{\mu\nu} A^{\nu\mu}$, since $\omega_{\mu\nu}$ is antisymmetric):

$$i\omega_{\mu\nu}[J^{\mu\nu}, J^{\sigma\rho}] = 2\left(\eta^{\sigma\nu}\omega_{\mu\nu} J^{\mu\rho} + \eta^{\rho\nu}\omega_{\mu\nu} J^{\sigma\mu}\right) = \omega_{\mu\nu}(\eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\sigma} J^{\nu\rho} + \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\rho} J^{\mu\sigma}),$$
$$-i\epsilon_\mu [P^\mu, J^{\sigma\rho}] = -\eta^{\sigma\mu}\epsilon_\mu P^\rho + \eta^{\rho\mu}\epsilon_\mu P^\sigma,$$
$$i\omega_{\mu\nu}[J^{\mu\nu}, P^\sigma] = 2\eta^{\sigma\nu}\omega_{\mu\nu} P^\mu = \omega_{\mu\nu}(\eta^{\sigma\nu} P^\mu - \eta^{\sigma\mu} P^\nu),$$
$$-i\epsilon_\mu [P^\mu, P^\sigma] = 0.$$

The second and third of those four equations are trivially equivalent, thus only the following three equations remain and make up the *Lie algebra of Poincaré group*:

$$i[J^{\mu\nu}, J^{\sigma\rho}] = \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\sigma} J^{\nu\rho} + \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\rho} J^{\mu\sigma},$$
$$i[P^\mu, J^{\sigma\rho}] = \eta^{\sigma\mu} P^\rho - \eta^{\rho\mu} P^\sigma,$$
$$i[P^\mu, P^\sigma] = 0.$$

3 Classical Field Theory

3.1 The Lagrangian Density and Euler-Lagrange Equations

3.1.1 The Euler-Lagrange Equations

In contrast to classical mechanics, classical *field* theory is based on a Lagrangian *density* $\mathcal{L}$ rather than a Lagrangian L, although they are closely related by

$$L = \int d^3x\, \mathcal{L}.$$

The dynamics of fields can be deduced by the variational principle applied to an action functional, involving a Lagrange density $\mathcal{L}$, which is a function of fields $\phi_a(x)$, $a = 1, 2, 3, \ldots$ and its derivatives:

$$S(\Omega) = \int dt\, L = \int_\Omega d^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a),$$

where Ω is a subset of the spacetime $\mathcal{M}$ and we assumed that $\mathcal{L}$ is not a function of higher derivatives of ϕ_a. Typically, we have $\Omega = \mathcal{M}$. We assume now that S is stationary under any small variations of the fields $\phi_a \to \phi_a + \delta\phi_a$ and that those variations vanish on the boundary $\partial\Omega$. The variation of S now reads

$$\delta S = \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right) = \int_\Omega d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \delta\phi_a.$$

The condition for ϕ_a to be the physical fields with the correct dynamics is that δS needs to vanish for *any* $\delta\phi_a$; this is what we meant when we assumed S to be stationary. Obviously, $\delta S = 0\ \forall \delta\phi_a$ is equivalent to

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0,$$

called the *Euler-Lagrange equations*.

3.1.2 The Lagrangians of the Klein-Gordon and the Dirac Field

Any well-known equation of motion can be encrypted in a Lagrangian $\mathcal{L}$. For example,

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 - \frac{1}{2} m^2 \phi^2, \qquad (\partial^\mu \phi)^2 \coloneqq (\partial^\mu \phi)(\partial_\mu \phi)$$

(sum over μ is implied) gives the Klein-Gordon equation

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -m^2\phi - \partial_\mu \partial^\mu \phi \qquad \Leftrightarrow \qquad (\Box + m^2)\phi(x) = 0$$

when plugged into the Euler-Lagrange equations. Here, we used the *quabla* operator $\Box \coloneqq \partial^\mu \partial_\mu$.

For a complex Klein-Gordon field, we should use the Lagrangian

$$\mathcal{L} = |\partial^\mu \phi|^2 - m^2 |\phi|^2 = (\partial^\mu \phi)^* (\partial_\mu \phi) - m^2 \phi \phi^*,$$

and treat ϕ and ϕ^* as independent fields, which gives the Euler-Lagrange equations

$$0 = \frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = -m^2\phi - \partial_\mu \partial^\mu \phi \qquad \Leftrightarrow \qquad (\Box + m^2)\phi(x) = 0,$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = -m^2\phi^* - \partial_\mu\partial^\mu\phi^* \qquad \Leftrightarrow \qquad (\Box + m^2)\phi^*(x) = 0.$$

If we take the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

and treat $\bar{\psi} = \psi^\dagger\gamma^0$ as an independent field, there are two Euler-Lagrange equations, namely

$$0 = \frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = (i\not{\partial} - m)\psi - 0,$$
$$0 = \frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = -m\bar{\psi} - \partial_\mu\bar{\psi}i\gamma^\mu = -\bar{\psi}(i\overleftarrow{\not{\partial}} + m).$$

Here we used the following definition of a derivative acting to the left:

$$f(x)\overleftarrow{\partial}_\mu g(x) := \left(\partial^\mu f(x)\right) g(x).$$

That is, $\overleftarrow{\partial}_\mu$ only acts on f, but not on g. For our Dirac equation, this notation turns out to be handy, since the alternative equivalent would be rather ugly, because we cannot commute $\bar{\psi}$ with the γ matrix inside $\overleftarrow{\not{\partial}}$:

$$0 = \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = (\partial_\mu\bar{\psi})i\gamma^\mu + \bar{\psi}m.$$

3.2 Noether's Theorem

3.2.1 Variation of the Integration Measure

We know that differentials like d^4x transform under coordinate transformations by a Jacobian determinant as follows:

$$d^4x' = d^4x \det[\partial x'^\nu/\partial x^\mu],$$

where $[\partial x'^\nu/\partial x^\mu]$ is supposed to denote the matrix with elements $\partial x'^\nu/\partial x^\mu$:

$$\left[\frac{\partial x'^\nu}{\partial x^\mu}\right]_{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\beta}.$$

We consider a coordinate transformation of the form $x'^\mu = x^\mu + \delta x^\mu \Leftrightarrow \delta x^\mu = x'^\mu - x^\mu$, thus also the variation of the differential obeys $\delta d^4x = d^4x' - d^4x$. We can write this variation, using the Jacobian, as

$$\delta d^4x = d^4x' - d^4x = d^4x\left(\det\frac{\partial x'^\nu}{\partial x^\mu} - 1\right) = d^4x\left(\det\left[\partial_\mu x^\nu + \partial_\mu\delta x^\nu\right] - 1\right)$$
$$= d^4x\left(\det\left[\delta^\nu_\mu + \partial_\mu\delta x^\nu\right] - 1\right).$$

For $a_{\mu\nu} := \partial_\mu\delta x^\nu$, this determinant is of the form

$$\det\left[\delta^\nu_\mu + \partial_\mu\delta x^\nu\right] = \det\begin{pmatrix} 1+a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & 1+a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & 1+a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & 1+a_{44} \end{pmatrix} = 1 + \sum_i a_{ii} + \mathcal{O}(a^2).$$

Since δx^ν is infinitesimal, we find

$$\delta d^4x = d^4x\left(1 + \partial_\mu\delta x^\mu - 1\right) = d^4x\, \partial_\mu\delta x^\mu.$$

3.2.2 Variation of the Fields

Let us plug in a zero into our variation rule for the fields:

$$\phi'_a(x') = \phi_a(x) + \delta\phi_a(x) \qquad \Leftrightarrow \qquad \delta\phi_a(x) = \phi'_a(x') \underbrace{- \phi'_a(x) + \phi'_a(x)}_{=0} - \phi_a(x).$$

The first two terms of this expression are the difference of a function ϕ'_a between two nearby points x' and x, which is generally is given by its derivative times the separation:

$$\phi'_a(x') - \phi'_a(x) = \left(\partial_\mu \phi'_a(x)\right)(x'^\mu - x^\mu) = \delta x^\mu\, \partial_\mu \phi'_a(x) = \delta x^\mu\, \partial_\mu \phi_a(x).$$

In the last step, we used $\phi'_a = \phi_a + \mathcal{O}(\delta\phi_a)$. Since this term only contains the infinitesimal δx^μ, we could neglected correction of order $\delta x^\mu\, \delta\phi_a$.

Let us define the last two terms of the expression for $\delta\phi_a$ as the "simple" variation δ_0, that varies the fields but not the coordinates:

$$\delta_0\phi_a(x) \coloneqq \phi'_a(x) - \phi_a(x).$$

Then, we can write

$$\delta\phi_a(x) = \delta x^\mu\, \partial_\mu \phi_a(x) + \delta_0\phi_a(x) \qquad \Leftrightarrow \qquad \delta_0\phi_a(x) = \delta\phi_a(x) - \delta x^\mu\, \partial_\mu \phi_a(x).$$

3.2.3 Variation of the Lagrangian

The derivation of the Lagrangian is understood to be defined as

$$\delta\mathcal{L}(\phi_a(x), \dots) = \mathcal{L}(\phi'_a(x'), \dots) - \mathcal{L}(\phi_a(x), \dots)$$

(obviously, there is nothing like a dashed Lagrangian $\mathcal{L}'$, since we only vary coordinates and fields, but certainly not the Lagrangians themselves). The dots … stand for the derivatives of the fields; we omit to denote them explicitly in this computation.

Just as for the fields in (>3.2.2), let us plug in a zero into the equation above:

$$\delta\mathcal{L}(\phi_a(x), \dots) = \mathcal{L}(\phi'_a(x'), \dots) \underbrace{- \mathcal{L}(\phi'_a(x), \dots) + \mathcal{L}(\phi'_a(x), \dots)}_{=0} - \mathcal{L}(\phi_a(x), \dots).$$

In the end, $\mathcal{L}(\phi'_a(x), \dots)$ is a function of x. Thus, the first two terms describe a function evaluated at x' minus the same function at x. Just as in (>3.2.2), such a difference is the same as the derivative of the function at x times the difference of $x' - x = \delta x$:

$$\mathcal{L}(\phi'_a(x'), \dots) - \mathcal{L}(\phi'_a(x), \dots) = \delta x^\mu\, \partial_\mu \mathcal{L}(\phi'_a(x), \dots) = \delta x^\mu\, \partial_\mu \mathcal{L}(\phi_a(x), \dots).$$

As in (>3.2.2), we used that ϕ'_a and ϕ_a only differ by an infinitesimal amount. Since this term already contains the infinitesimal δx^μ, we could replace ϕ'_a by ϕ_a to the order of one infinitesimal object.

The second two terms are the simple variation of the Lagrangian, $\delta_0\mathcal{L}$. We can expand $\delta_0\mathcal{L}$ in the simple derivations of the fields in the same way as we expanded $\delta\mathcal{L}$ in the deviations of the fields when we derived the Euler-Lagrange equations in (>3.1.1):

$$\mathcal{L}(\phi'_a(x), \dots) - \mathcal{L}(\phi_a(x), \dots) = \delta_0\mathcal{L}(\phi_a(x), \dots) = \overbrace{\frac{\partial\mathcal{L}}{\partial\phi_a}}^{=\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}} \delta_0\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta_0(\partial_\mu\phi_a)$$
$$= \left(\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) \delta_0\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} (\partial_\mu \delta_0\phi_a) = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta_0\phi_a\right).$$

Here we used, that we assumed that the fields ϕ_a obey the Euler-Lagrange equations.

Combining the results for the first and last two terms of $\delta\mathcal{L}$, we find

$$\delta\mathcal{L} = \delta x^\mu \, \partial_\mu \mathcal{L} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta_0 \phi_a \right),$$

we all fields and variables are undashed.

3.2.4 Combining the Results

Combining the result of (>3.2.1), (>3.2.2) and (>3.2.3), we find

$$\begin{aligned}
\delta(d^4x\,\mathcal{L}) &= (\delta d^4x)\,\mathcal{L} + d^4x\,(\delta\mathcal{L}) = \left(d^4x\,\partial_\mu \delta x^\mu\right)\mathcal{L} + d^4x\left(\delta x^\mu\,\partial_\mu\mathcal{L} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta_0\phi\right)\right) \\
&= d^4x\,\partial_\mu\left(\mathcal{L}\,\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta_0\phi\right) = d^4x\,\partial_\mu\left(\eta^{\mu\nu}\,\mathcal{L}\,\delta x_\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(\delta\phi_a - \delta x^\nu\,\partial_\nu\phi_a)\right) \\
&= d^4x\,\partial_\mu\left(\left(\eta^{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a\right)\delta x_\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) \\
&= d^4x\,\partial_\mu\left(-\mathcal{T}^{\mu\nu}\delta x_\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right).
\end{aligned}$$

Here, we defined the energy momentum tensor

$$\mathcal{T}^{\mu\nu} \coloneqq \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a - \eta^{\mu\nu}\mathcal{L}.$$

3.3 The Hamiltonian in Classical Field Theory

3.3.1 Hamiltonian of the Klein-Gordon Field

Using the Klein-Gordon Lagrangian from section 3.1,

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)^2 - \frac{m^2}{2}\phi^2 = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2,$$

we find

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}.$$

Thus,

$$\begin{aligned}
H &= \int d^3x\,\mathcal{H} = \int d^3x\,\mathcal{T}^{00} = \int d^3x\,\left(\Pi\dot{\phi} - \mathcal{L}\right) = \int d^3x\,\left(\Pi\dot{\phi} - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\right) \\
&= \frac{1}{2}\int d^3x\,\left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right).
\end{aligned}$$

3.3.2 Hamiltonian of the Dirac Field

Using the Dirac Lagrangian from section 3.1,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi,$$

we find

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \bar{\psi}i\gamma^0, \qquad \overline{\Pi} = \frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}} = 0.$$

Thus,

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, \mathcal{T}^{00} = \int d^3x\, (\Pi\dot{\psi} - \mathcal{L}) = \int d^3x\, (\bar{\psi} i\gamma^0 \dot{\psi} - \bar{\psi}(i\partial\!\!\!/ - m)\psi).$$

Noether's theorem assumes, that the fields obey their equations of motion. Thus, we can (or even need to) assume that ψ fulfills the Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$. Thus, we can drop the second term in the integral.

3.5 Global U(1) Symmetry yields Particle Currents

3.5.1 Current of the Klein-Gordon Field

Consider the complex Klein-Gordon field Lagrangian from section 3.1,

$$\mathcal{L} = |\partial^\mu \phi|^2 - m^2|\phi|^2$$

and the global U(1) gauge transformation $U = e^{iq\theta}$ for some real parameter θ. That is, we transform the fields ϕ and ϕ^* according to

$$\phi \to e^{iq\theta}\phi, \quad \phi^* \to e^{-iq\theta}\phi^*.$$

(note, that we now consider the case where U is no matrix and ϕ is no vector, thus the second transformation above is the equivalent to $\phi^\dagger \to \phi^\dagger U^\dagger$). Obviously, the Lagrangian is invariant under this transformation. Thus, there will be a conserved Noether current j^μ, see section 3.2. We did not transform the coordinates, thus $\delta x^\mu = 0$, but we did transform the fields. Expanding for infinitesimal θ, we find $\phi \to \phi + \delta\phi$ with $\delta\phi = iq\theta\phi$. Thus, using the formula from section 3.2, we find (with the independent fields $\phi_{a=1} = \phi$ and $\phi_{a=2} = \phi^*$)

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\delta \phi_a}{\delta\theta} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\delta\theta} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{\delta \phi^*}{\delta\theta} = (\partial^\mu\phi^*)(iq\phi) + (\partial^\mu\phi)(-iq\phi^*) \\ &= iq(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi). \end{aligned}$$

3.5.2 Current of the Dirac Field

The Dirac Lagrangian from section 3.1 reads

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi.$$

It is trivially invariant under *global* U(1) transformation $\psi \to e^{iq\theta}\psi, \bar{\psi} \to \bar{\psi}e^{-iq\theta}$ (that is, $\delta\psi = iq\theta\psi$) Thus, there is a Noether's current that reads

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \frac{\delta\psi}{\delta\theta} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}}_{=0} \frac{\delta\bar{\psi}}{\delta\theta} = (\bar{\psi} i\gamma^\mu)(iq\psi) = -q\bar{\psi}\gamma^\mu\psi.$$

Noether's theorem tells us, that $\partial_\mu j^\mu = 0$. Obviously, we can also choose the current $j^\mu = q\bar{\psi}\gamma^\mu\psi$ (with positive sign), since it also obeys this relation.

3.6 Electrodynamics

3.6.1 Maxwell Equations as Euler-Lagrange Equations

Plugging in the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j_\mu A^\mu$$

into the Euler-Lagrange equations, we find the Maxwell equations in covariant formulation:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} &= -j_\mu + \frac{1}{2}\partial^\nu \left((\partial_\sigma A_\kappa - \partial_\kappa A_\sigma) \frac{\partial}{\partial(\partial^\nu A^\mu)} (\partial^\sigma A^\kappa - \partial^\kappa A^\sigma) \right) \\
&= -j_\mu + \frac{1}{2}\partial^\nu \left((\partial_\sigma A_\kappa - \partial_\kappa A_\sigma)(\delta^\sigma_\nu \delta^\kappa_\mu - \delta^\kappa_\nu \delta^\sigma_\mu) \right) \\
&= -j_\mu + \frac{1}{2}\partial^\nu \left((\partial_\nu A_\mu - \partial_\mu A_\nu) - (\partial_\mu A_\nu - \partial_\nu A_\mu) \right) = -j_\mu + \partial^\nu F_{\nu\mu} \stackrel{!}{=} 0 \\
\Leftrightarrow \quad \partial_\nu F^{\nu\mu} &= j^\mu.
\end{aligned}$$

Note that in the first step, we used the indices symmetry for the product rule like

$$\frac{1}{4}\partial_\mu B^\sigma B_\sigma = \frac{1}{4}\left((\partial_\mu B^\sigma) B_\sigma + B^\sigma (\partial_\mu B_\sigma) \right) = \frac{1}{4}\left((\partial_\mu B_\sigma) B^\sigma + B_\sigma (\partial_\mu B^\sigma) \right) = \frac{1}{2} B_\sigma (\partial_\mu B^\sigma).$$

3.6.2 Local U(1) Gauge Invariance of the QED Lagrangian

As we learned in section 3.4, *local* U(1) gauge invariance means invariance under the transformation

$$\psi(x) \to e^{iq\theta(x)}\, \psi(x), \qquad \bar\psi(x) \to \bar\psi(x)\, e^{-iq\theta(x)}.$$

Although we did not yet find the general transformation rule for the gauge field A^μ, the U(1) case corresponds to the well-known case of the gauge transformation that we already encountered long ago in electrodynamics:

$$A^\mu(x) \to A^\mu(x) - \partial^\mu \theta(x).$$

Let us consider the behavior of the different parts of the QED Lagrangian under this transformation. First, the kinetic term of the gauge fields contains the electromagnetic tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \to \partial^\mu (A^\nu - \partial^\nu \theta) - \partial^\nu (A^\mu - \partial^\mu \theta) = F^{\mu\nu}.$$

It is quite trivially invariant under the transformation. Thus, also the kinetic term $-F_{\mu\nu}F^{\mu\nu}/4$ is invariant.

Let us further consider what happens to the covariant derivative:

$$\begin{aligned}
D^\mu = \partial^\mu + iqA^\mu \to \partial^\mu + iq(A^\mu - \partial^\mu\theta) &= \partial^\mu - iq\partial^\mu\theta + iqA^\mu = e^{iq\theta}\partial^\mu e^{-iq\theta} + iqA^\mu \\
&= e^{iq\theta}(\partial^\mu + iqA^\mu)e^{-iq\theta} = e^{iq\theta} D^\mu e^{-iq\theta} = U D^\mu U^\dagger.
\end{aligned}$$

Thus, also the Dirac term – but only together with the interaction – is gauge invariant under local U(1):

$$\bar\psi(i\not{D} - m)\psi \to \bar\psi e^{-iq\theta}\left(i e^{iq\theta} \not{D} e^{-iq\theta} - m\right) e^{iq\theta}\psi = \bar\psi(i\not{D} - m)\psi.$$

3.7 Non-Abelian Gauge Theories

3.7.1 The Dirac Part

We now want to find a Lagrangian invariant under a local SU(N) transformation. Let us use the QED Lagrangian as a starting point, which is invariant under local U(1) transformations, as we found in (>3.6.2). The Dirac part of the QED Lagrangian reads

$$\mathcal{L} = \bar\psi(i\not{D} - m)\psi, \qquad D_\mu = \partial_\mu + igA_\mu.$$

If the covariant derivative and the fields transform according to

$$D_\mu \to D'_\mu = U D_\mu U^\dagger, \qquad \psi \to \psi' = U\psi, \qquad \bar\psi \to \bar\psi' = \bar\psi U^\dagger, \qquad U = e^{i\theta^a t^a},$$

the Dirac part of the Lagrangian is obviously invariant.

In order words: We want to keep the neat transformation rules above also in the SU(N) case, but we are willing to accept modifications/ generalizations to the transformation rule for the gauge fields A^μ

in order to find a SU(N) gauge invariant Lagrangian. Or in yet other words: We could also try to keep the transformation rule for the gauge field $A^\mu \to A^\mu - \partial^\mu \theta$ and try to adapt the other transformations rules to make the Lagrangian SU(N) invariant; however, this approach would not lead us to results that are observed in nature.

Plugging in the definition of the covariant derivative, we find the transformation rule for the gauge field A_μ, that is necessary for the Dirac part to be SU(N) gauge invariant:

$$D'_\mu \stackrel{!}{=} U D_\mu U^\dagger \qquad \Leftrightarrow \qquad \partial_\mu + igA'_\mu \stackrel{!}{=} U\big(\partial_\mu + igA_\mu\big)U^\dagger$$

$$\Leftrightarrow \qquad igA'_\mu \stackrel{!}{=} U\partial_\mu U^\dagger - \partial_\mu + igUA_\mu U^\dagger.$$

Note, that D_μ is an operator, acting on something to the right. Let us call this something $f(x)$. Then,

$$\big(U\partial_\mu U^\dagger - \partial_\mu\big)f(x) = \big(U\big(\partial_\mu U^\dagger\big) + UU^\dagger\partial_\mu - \partial_\mu\big)f(x) = U\big(\partial_\mu U^\dagger\big)f(x)$$

and hence

$$A'_\mu = \frac{1}{ig}U\big(\partial_\mu U^\dagger\big) + UA_\mu U^\dagger.$$

Recall that we argued in section 3.4, that ψ must a be a vector if it transforms like $\psi \to U\psi$ and if U is a matrix, because $U\psi$ plays the same role as ψ and must therefore be of the same structure. The same holds for the gauge fields A_μ. However, the expression above only makes sense, if A_μ and A'_μ are *matrices.* Using $U = e^{i\theta^a t^a} = 1 + i\theta^a t^a + \mathcal{O}(\theta^2)$, we find

$$\begin{aligned} A'_\mu &= \frac{1}{ig}(1 + i\theta^a t^a)\left(\partial_\mu\big((1 - i\theta^a t^a)\big)\right) + (1 + i\theta^a t^a)A_\mu(1 - i\theta^a t^a) + \mathcal{O}(\theta^2) \\ &= A_\mu - \frac{1}{g}\big(\partial_\mu\theta^a\big)t^a + i\theta^a\big[t^a, A_\mu\big] + \mathcal{O}(\theta^2). \end{aligned}$$

Since A_μ is a matrix, it does not necessarily commute with t^a. Ignoring the third term, this transformation is an addition (or subtraction) of a term proportional to t^a. Thus, it is reasonable to assume that A_μ is a linear combination of t^a, that is

$$A_\mu = A_\mu^a t^a,$$

where a sum over a is implied and $A_\mu^a(x)$ are the coefficients of the constant matrices t^a. This proposal is even more convincing, when we find out that for this construction, the third term is *also* proportional to t^a:

$$A'^a_\mu t^a = A^a_\mu t^a - \frac{1}{g}\big(\partial_\mu\theta^a\big)t^a + i\theta^a \underbrace{\big[t^a, A^b_\mu t^b\big]}_{=A^b_\mu i f^{abc} t^c} = A^a_\mu t^a - \frac{1}{g}\big(\partial_\mu\theta^a\big)t^a - \theta^c A^b_\mu \underbrace{f^{cba}}_{=-f^{abc}} t^a$$

$$\Leftrightarrow \qquad A'^a_\mu = A^a_\mu - \frac{1}{g}\partial_\mu\theta^a + f^{abc}A^b_\mu\theta^c.$$

3.7.2 Electromagnetic Field Tensor

In section 3.6, we found that we can write the electromagnetic field tensor in the context of electrodynamics as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{1}{iq}[D^\mu, D^\nu].$$

Since the behavior under SU(N) transformation of the covariant derivative D_μ is much nicer than that of the fields A^a_μ, it is natural to adopt the second rather than the first expression as our new definition in the context of SU(N) invariant theories. Using $D_\mu = \partial_\mu + igA^a_\mu t^a$, we find

$$
\begin{aligned}
F_{\mu\nu} &:= \frac{1}{ig}\left[D_\mu, D_\nu\right] = \frac{1}{ig}\left[\partial_\mu + igA_\mu^a t^a, \partial_\nu + igA_\nu^b t^b\right] \\
&= \frac{1}{ig}\left(\left[\partial_\mu, igA_\nu^b t^b\right] + \left[igA_\mu^a t^a, \partial_\nu\right] + \left[igA_\mu^a t^a, igA_\nu^b t^b\right]\right) \\
&= \left[\partial_\mu, A_\nu^b\right]t^b + \left[A_\mu^a, \partial_\nu\right]t^a + igA_\mu^a A_\nu^b[t^a, t^b] = \left(\partial_\mu A_\nu^b\right)t^b - \left(\partial_\nu A_\mu^a\right)t^a - gA_\mu^a A_\nu^b f^{abc} t^c.
\end{aligned}
$$

If we expand $F_{\mu\nu} = F_{\mu\nu}^a t^a$, we find

$$
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gA_\mu^b A_\nu^c f^{abc}.
$$

Since the transformation of the covariant derivative under $\mathrm{SU}(N)$ is $D_\mu \to UD_\mu U^\dagger$, the electromagnetic field tensor transforms according to

$$
F_{\mu\nu} = \frac{1}{iq}[D^\mu, D^\nu] \to \frac{1}{iq}[UD^\mu U^\dagger, UD^\nu U^\dagger] = \frac{1}{iq}U[D^\mu, D^\nu]U^\dagger = UF_{\mu\nu}U^\dagger.
$$

However, in contrast to QED, $F_{\mu\nu}$ is a matrix now that does not commute with U. Hence, $F_{\mu\nu}$ alone is *not* invariant under $\mathrm{SU}(N)$.

4 QUANTIZED KLEIN-GORDON FIELD

4.2 Lorentz Invariant Phase Space Measure

4.2.1 Lorentz Invariant Phase Space Measure

When we integrate over the energy-momentum space d^4p, we actually only want to integrate over such four-vectors p^μ, which obey $p^2 = m^2$ or equivalently $p_0^2 = \vec{p}^2 + m^2$. This condition fixes the energy component p_0 for a given three-momentum $\vec{p}$ and we should be able to use this condition to turn the d^4p integral into a d^3p integral. Since we expect that this is possible, we use the measure $d^4p/(2\pi)^3$ instead of $d^4p/(2\pi)^4$ (see also the footnote on page 21).

Let us try to impose the relation $p^2 = m^2$ using a δ-function. That is, we want to use the integration measure

$$\frac{d^4p}{(2\pi)^3}\delta(p^2 - m^2) = \frac{d^4p}{(2\pi)^3}\delta(p_0^2 - \vec{p}^2 - m^2),$$

ensuring that only momenta with $p^2 = m^2$ are covered by the integral. Let $\omega_p^2 \coloneqq \vec{p}^2 + m^2$. Then, using properties of the δ-function, we find

$$\frac{d^4p}{(2\pi)^3}\delta\left(p_0^2 - \omega_p^2\right) = \frac{d^4p}{(2\pi)^3}\frac{1}{2|p_0|}\sum_{\pm}\delta\left(p_0 \pm \omega_p\right).$$

Since energies should be positive, we are only interested into the solution $p_0 = +\omega_p$. That is, the condition $p^2 - m^2$ still allows negative energies and we want to impose the additional condition $p_0 > 0$, which we can do using a θ-function:

$$d\tilde{p} \coloneqq \frac{d^4p}{(2\pi)^3}\delta(p^2 - m^2)\theta(p_0) = \frac{d^4p}{(2\pi)^3}\frac{1}{2|p_0|}\sum_{\pm}\delta\left(p_0 \pm \omega_p\right)\theta(p_0) = \frac{d^4p}{(2\pi)^3}\frac{1}{2|p_0|}\delta\left(p_0 - \omega_p\right)$$
$$= \frac{d^3p}{(2\pi)^3 2\omega_p}.$$

Note, that the measure $d\tilde{p}$ is Lorentz invariant for the following reason: When we consider the form of $d\tilde{p}$ in terms of $\sim d^4p\,\delta(p^2 - m^2)\theta(p_0)$, we see immediately that d^4p and p^2 are Lorentz invariant and $\theta(p_0)$ is Lorentz invariant under usual Lorentz transformation (not time inversion, but boosts and rotations). Thus, $d\tilde{p}$ is Lorentz invariant.

4.4 Quantization of the Real Klein-Gordon Field

4.4.1 Analogy: Coupled Harmonic Oscillators in Ordinary Quantum Mechanics

Suppose we have a coupled harmonic oscillator Hamiltonian of the form

$$H = \sum_i \frac{p_i^2}{2m} + \frac{m}{2}q_i Q_{ij} q_j$$

where $Q_{ij} = Q_{ji}$. After quantization, the q's and p's have become operators. Symmetric matrices like Q can always be diagonalized by orthogonal matrices O (with $O^T O = \mathbb{I}$):

$$D = OQO^T.$$

We introduce q_i' and p_i' as

$$q_i = O_{ij}q_j', \quad p_i = O_{ij}p_j' \quad \Leftrightarrow \quad q_j' = O_{ij}^T q_i = O_{ji}q_i, \quad p_j' = O_{ij}^T p_i = O_{ji}p_i,$$

where we also need to transform the $\hat{p}$'s to maintain the commutation relations:

$$[q_i', p_j'] = [O_{ik}q_k, O_{jl}p_l] = O_{ik}O_{jl}[q_k, p_l] = iO_{ik}O_{jl}\delta_{kl} = iO_{ik}O_{jk} = iO_{ik}O_{kj}^T = i\delta_{ij}.$$

If we substitute those new variables into H we find

$$\begin{aligned} H &= \frac{1}{2m}p_ip_i + \frac{m}{2}q_iQ_{ij}q_j = \frac{1}{2m}O_{ji}p_j'O_{ki}p_k' + \frac{m}{2}O_{ki}q_k'Q_{ij}O_{jl}^Tq_l' = \frac{1}{2m}p_i'p_i' + \frac{m}{2}q_k'D_{kl}q_l' \\ &= \sum_i \left(\frac{p_i'^2}{2m} + \frac{m\omega_i^2}{2}q_i'^2\right), \end{aligned}$$

where we used that $D_{kl} = \omega_k^2\delta_{kl}$ is diagonal, and we have our diagonalized Hamiltonian. This can now be solved by ladder operators $a_i, a_i^\dagger$ introduced like *(no sum convention!)*

$$q_i' = A_i(a_i^\dagger + a_i), \quad p_i' = iB_i(a_i^\dagger - a_i)$$

$$\Leftrightarrow \quad a_i = \frac{1}{2A_iB_i}(B_iq_i' + iA_ip_i'), \quad a_i^\dagger = \frac{1}{2A_iB_i}(B_iq_i' - iA_ip_i'),$$

and from $[q_i, p_j] = i\delta_{ij}$ we find that

$$\begin{aligned} [a_i, a_j^\dagger] &= \left[\frac{1}{2A_iB_i}(B_iq_i' + iA_ip_i'), \frac{1}{2A_jB_j}(B_jq_i' - iA_jp_i')\right] = \frac{1}{4}\left(-\frac{i}{A_iB_j}[q_i', p_j'] + \frac{i}{A_jB_i}[p_i', q_j']\right) \\ &= \frac{1}{2A_iB_i}\delta_{ij}. \end{aligned}$$

For simplicity, we demand $[a_i, a_j^\dagger] = \delta_{ij}$, thus the ladder operators need to be dimensionless. We therefore take,

$$A_i = \frac{1}{\sqrt{2m\omega_i}}, \quad B_i = \sqrt{\frac{m\omega_i}{2}},$$

they do the job. In terms of those, the Hamiltonian can be written as

$$\begin{aligned} H &= \sum_i \frac{\omega_i}{4}\left(-(a_i^\dagger + a_i)^2 + (a_i^\dagger + a_i)^2\right) = \sum_i \frac{\omega_i}{2}(a_ia_i^\dagger + a_i^\dagger a_i) = \sum_i \frac{\omega_i}{2}(2a_i^\dagger a_i + [a_i, a_i^\dagger]) \\ &= \sum_i \omega_i\left(a_ia_i^\dagger + \frac{1}{2}\right). \end{aligned}$$

4.4.2 Analogous Derivation of the Hamiltonian of the Klein-Gordon Field

The Hamiltonian of the real Klein-Gordon field was found to be

$$H = \frac{1}{2}\int d^3x \left(\Pi^2(\vec{x}) + (\nabla\phi(\vec{x}))^2 + m^2\phi^2(\vec{x})\right).$$

We now also want to diagonalize this Hamiltonian. We do this quite analogous to the harmonic oscillators in QM: q_i is now $\phi(\vec{x})$ and p_i is $\Pi(\vec{x})$, the indices i are the variables $\vec{x}$, thus summation becomes integration. The analog of $q_i = O_{ij}q_j'$ is

$$\phi(\vec{x}) = \int d^3p\, K(\vec{x}, \vec{p})\phi(\vec{p}), \quad \Pi(\vec{x}) = \int d^3p\, K(\vec{x}, \vec{p})\Pi(\vec{p}).$$

where $K(\vec{x}, \vec{p})$ plays the role of O_{ij}^T and the fields in dependence of $\vec{p}$ play the role of the dashed coordinates. We know that $O_{ij}O_{ik}^T = \delta_{jk}$, thus we want something like

$$\int d^3x\, K(\vec{x}, \vec{p})K^\dagger(\vec{x}, \vec{p}') \sim \delta(\vec{p} - \vec{p}').$$

The first and most elegant thing coming to mind is $K(\vec{x},\vec{p}) = e^{i\vec{x}\vec{p}}/(2\pi)^3$; in that case, $\phi(\vec{p})$ is the Fourier transform of $\phi(\vec{x})$.[1] We will use the short-hand notation $d^n\bar{p} \coloneqq d^n p/(2\pi)^n$. However, it is very important to note, that in contrast to q_i' and p_i', those $\phi(p)$ and $\Pi(\vec{p})$ are *not* Hermitian:

$$\phi^\dagger(\vec{x}) = \int d^3\bar{p}\, e^{-i\vec{x}\vec{p}}\phi^\dagger(\vec{p}) = \int d^3\bar{p}\, e^{i\vec{x}\vec{p}}\phi^\dagger(-\vec{p}) \stackrel{!}{=} \int d^3\bar{p}\, e^{i\vec{x}\vec{p}}\phi(\vec{p}) = \phi(\vec{x})$$
$$\Leftrightarrow \quad \phi^\dagger(\vec{p}) = \phi(-\vec{p}), \quad \Pi^\dagger(\vec{p}) = \Pi(-\vec{p}).$$

As in QM case, we want the commutator relation to stay the same for the $\phi(\vec{p}), \Pi(\vec{p})$ as for the $\phi(\vec{x}), \Pi(\vec{x})$. To accomplish that, we need to add a $\dagger$ to one of the operators in the commutator. Since q_i' and p_i' where Hermitian and so are $\phi(\vec{x}), \Pi(\vec{x})$, this can still be viewed as analogous to the QM case:

$$[\phi(\vec{p}), \Pi^\dagger(\vec{p}')] = \left[\int d^3x\, e^{-i\vec{x}\vec{p}}\phi(\vec{x}), \int d^3y\, e^{i\vec{y}\vec{p}'}\Pi(\vec{y})\right] = \int d^3x\, d^3y\, e^{-i\vec{x}\vec{p}}e^{i\vec{y}\vec{p}'}[\phi(\vec{x}), \Pi(\vec{y})]$$
$$= i\int d^3x\, d^3y\, e^{-i\vec{x}\vec{p}}e^{i\vec{y}\vec{p}'}\delta(\vec{x}-\vec{y}) = i\int d^3x\, e^{-i\vec{x}(\vec{p}-\vec{p}')} = (2\pi)^3\, i\delta(\vec{p}-\vec{p}').$$

Well, it's the same up to a factor, but alright. Obviously, $[\phi^\dagger(p), \Pi(\vec{p}')]$ will have the same result. But even more importantly, they should also diagonalize the Hamiltonian. The Π^2 part and the ϕ^2 part behave equivalently:

$$\int d^3x\, \Pi^2(\vec{x}) = \int d^3x\, d^3\bar{p}\, d^3\bar{p}'\, e^{i\vec{x}(\vec{p}+\vec{p}')}\Pi(\vec{p})\Pi(\vec{p}') = \int d^3x\, d^3\bar{p}\, d^3\bar{p}'\, e^{-i\vec{x}(\vec{p}-\vec{p}')}\Pi^\dagger(\vec{p})\Pi(\vec{p}'),$$

where we turned the $\vec{p} \to -\vec{p}$ and used $\Pi^\dagger(\vec{p}) = \Pi(-\vec{p})$. The derivative term reads in the same way

$$\int d^3x\, \left(\nabla\phi(\vec{x})\right)^2 = \int d^3x\, d^3\bar{p}\, d^3\bar{p}' \left(\nabla e^{i\vec{x}\vec{p}}\phi(\vec{p})\right)\left(\nabla e^{i\vec{x}\vec{p}'}\phi(\vec{p}')\right)$$
$$= \int d^3x\, d^3\bar{p}\, d^3\bar{p}' \left(\nabla e^{-i\vec{x}\vec{p}}\phi^\dagger(\vec{p})\right)\left(\nabla e^{i\vec{x}\vec{p}'}\phi(\vec{p}')\right)$$
$$= \int d^3x\, d^3\bar{p}\, d^3\bar{p}' \left(-i\vec{p}e^{-i\vec{x}\vec{p}}\phi^\dagger(\vec{p})\right)\left(i\vec{p}'e^{i\vec{x}\vec{p}'}\phi(\vec{p}')\right)$$
$$= \int d^3x\, d^3\bar{p}\, d^3\bar{p}'\, e^{-i\vec{x}(\vec{p}-\vec{p}')}\vec{p}\vec{p}'\phi^\dagger(\vec{p})\phi(\vec{p}').$$

Altogether, we have

[1] About the factor $(2\pi)^3$: For Fourier transformations we will use the convention

$$f(\vec{x}) = \int \underbrace{\frac{d^3p}{(2\pi)^3}}_{=:d^3\bar{p}} \hat{f}(\vec{p})e^{i\vec{p}\vec{x}}, \quad \hat{f}(\vec{p}) = \int d^3x\, f(\vec{x})e^{-i\vec{p}\vec{x}}.$$

We know that

$$\int d^3x\, e^{-i\vec{p}\vec{x}} \sim \delta(\vec{p}),$$

since for $\vec{p} = 0$ the integral is infinite and for $\vec{p} \neq 0$ the integral adds up all points on a unit circle in the complex plane, which gives zero. Thus, the Fourier transformation of 1 is proportional to the δ-function. Let us call the proportionality constant c and compute the inverse Fourier transformation:

$$\int d^3x\, e^{-i\vec{p}\vec{x}} = c\delta(\vec{p}) \quad \Rightarrow \quad 1 \stackrel{!}{=} \int d^3\bar{p}\, c\delta(\vec{p})\, e^{i\vec{p}\vec{x}} = \frac{c}{(2\pi)^3} \quad \Leftrightarrow \quad c = (2\pi)^3.$$

$$
\begin{aligned}
H &= \frac{1}{2}\int d^3x\left(\Pi^2(\vec{x}) + \left(\nabla\phi(\vec{x})\right)^2 + m^2\phi^2(\vec{x})\right)\\
&= \frac{1}{2}\int d^3x\, d^3\bar{p}\, d^3\bar{p}' e^{-i\vec{x}(\vec{p}-\vec{p}')}\left(\Pi^\dagger(\vec{p})\Pi(\vec{p}') + \vec{p}\vec{p}'\phi^\dagger(\vec{p})\phi(\vec{p}') + m^2\phi^\dagger(\vec{p})\phi(\vec{p}')\right)\\
&= \frac{1}{2}\int d^3\bar{p}\, d^3\bar{p}'\ (2\pi)^3\delta(\vec{p}-\vec{p}')\left(\Pi^\dagger(\vec{p})\Pi(\vec{p}') + \vec{p}\vec{p}'\phi^\dagger(\vec{p})\phi(\vec{p}') + m^2\phi^\dagger(\vec{p})\phi(\vec{p}')\right)\\
&= \frac{1}{2}\int d^3\bar{p}\ (|\Pi(\vec{p})|^2 + \vec{p}^2|\phi(\vec{p})|^2 + m^2|\phi(\vec{p})|^2) = \frac{1}{2}\int d^3\bar{p}\ (|\Pi(\vec{p})|^2 + (m^2+\vec{p}^2)|\phi(\vec{p})|^2)\\
&= \frac{1}{2}\int d^3\bar{p}\ \left(|\Pi(\vec{p})|^2 + \omega_p^2|\phi(\vec{p})|^2\right),
\end{aligned}
$$

where we defined

$$\omega_p^2 \coloneqq m^2 + \vec{p}^2, \quad |\phi|^2 \coloneqq \phi^\dagger\phi.$$

This Hamiltonian is now diagonalized, exactly like in the ordinary QM case.

The next step is to construct ladder operators. In analogy to the ordinary quantum mechanics case, the relation should be

$$\phi(\vec{p}) \sim \left(a_p^\dagger + a_p\right), \quad \Pi(\vec{p}) \sim i\left(a_p^\dagger - a_p\right).$$

However, this cannot be right, because the right-hand side *is* Hermitian, whereas we already saw, that $\phi(\vec{p})$ and $\Pi(\vec{p})$ are not: $\phi^\dagger(\vec{p}) = \phi(-\vec{p}), \Pi^\dagger(\vec{p}) = \Pi(-\vec{p})$. We need to fix this. How about

$$\phi(\vec{p}) = \frac{1}{2\omega_p}\left(e^{i\omega_p t}a_p^\dagger + e^{-i\omega_p t}a_p\right) \quad \Longrightarrow \quad \phi^\dagger(\vec{p}) = \frac{1}{2\omega_p}\left(e^{-i\omega_p t}a_p + e^{i\omega_p t}a_p^\dagger\right)?$$

This is still Hermitian, doesn't work. Let's try this:

$$
\begin{aligned}
\phi(\vec{p}) &= \frac{1}{2\omega_p}\left(e^{i\omega_p t}a_{-p}^\dagger + e^{-i\omega_p t}a_p\right) \quad &\Longrightarrow \quad \phi^\dagger(\vec{p}) &= \frac{1}{2\omega_p}\left(e^{-i\omega_p t}a_{-p} + e^{i\omega_p t}a_p^\dagger\right) = \phi(-\vec{p}).\\
\Pi(\vec{p}) &= \frac{i}{2}\left(e^{i\omega_p t}a_{-p}^\dagger - e^{-i\omega_p t}a_p\right) \quad &\Longrightarrow \quad \Pi^\dagger(\vec{p}) &= \frac{-i}{2}\left(e^{-i\omega_p t}a_{-p} - e^{i\omega_p t}a_p^\dagger\right) = \Pi(-\vec{p}).
\end{aligned}
$$

This works indeed![1] As in the quantum mechanics case, we chose our prefactors to get the desired commutator relation later. We can solve those equations for the ladder operators:

$$a_p = e^{i\omega_p t}\left(\omega_p\phi(\vec{p}) + i\Pi(\vec{p})\right), \quad a_{-p}^\dagger = e^{-i\omega_p t}\left(\omega_p\phi(\vec{p}) - i\Pi(\vec{p})\right).$$

Thus, the commutator reads

$$
\begin{aligned}
\left[a_p, a_{p'}^\dagger\right] &= \left[e^{i\omega_p t}\left(\omega_p\phi(\vec{p}) + i\Pi(\vec{p})\right), e^{-i\omega_p t}\left(\omega_p\phi(-\vec{p}') - i\Pi(-\vec{p}')\right)\right]\\
&= i\omega_p\left(-[\phi(\vec{p}),\Pi^\dagger(\vec{p}')] + [\Pi(\vec{p}),\phi^\dagger(\vec{p}')]\right) = (2\pi)^3 2\omega_p\delta(\vec{p}-\vec{p}').
\end{aligned}
$$

Plugging those equations into the Hamiltonian, it reads (recall that $|\Pi(\vec{p})|^2 \coloneqq \Pi^\dagger(\vec{p})\Pi(\vec{p})$)

$$
\begin{aligned}
H &= \frac{1}{2}\int d^3\bar{p}\ \left(|\Pi(\vec{p})|^2 + \omega_p^2|\phi(\vec{p})|^2\right) = \frac{1}{2}\int d^3\bar{p}\frac{1}{2}\left(a_{-p}a_{-p}^\dagger + a_p^\dagger a_p\right) = \frac{1}{2}\int d^3\bar{p}\ \frac{1}{2}\left(a_p a_p^\dagger + a_p^\dagger a_p\right)\\
&= \frac{1}{2}\int d^3\bar{p}\frac{1}{2}\left(2a_p^\dagger a_p + \left[a_p, a_p^\dagger\right]\right) = \frac{1}{2}\int d^3\bar{p}\ \left(a_p^\dagger a_p + \omega_p\delta(0)\right) = \int d\tilde{p}\ \omega_p\left(a_p^\dagger a_p + \omega_p\delta(0)\right),
\end{aligned}
$$

[1] Actually, the exponential factors are not really necessary, to make ϕ anti-Hermitian. At the end of this section there will be a comment about this issue.

where we used, that integration over $\vec{p}$ or $-\vec{p}$ is no different. We have now this infinite term $\sim \delta(0)$ appearing here. The workaround is to treat is as a constant and state that this constant always drops out, when we calculate differences in energies – for we can *measure* differences only.

Finally, we should address the question, how the actual fields $\phi(\vec{x}), \Pi(\vec{x})$ look like. We simply get them by the Fourier transformation of $\phi(\vec{p}), \Pi(\vec{p})$:

$$\begin{aligned}
\phi(\vec{x}) &= \int d^3\bar{p}\; e^{i\vec{p}\vec{x}}\phi(\vec{p}) = \int d\tilde{p}\;\left(e^{i(\omega_p t+\vec{p}\vec{x})}a_{-p}^\dagger + e^{-i(\omega_p t-\vec{p}\vec{x})}a_p\right) \\
&= \int d\tilde{p}\;\left(e^{i(\omega_p t-\vec{p}\vec{x})}a_p^\dagger + e^{-i(\omega_p t-\vec{p}\vec{x})}a_p\right) = \int d\tilde{p}\;\left(e^{ip\cdot x}a_p^\dagger + e^{-ip\cdot x}a_p\right), \\
\Pi(\vec{x}) &= \int d\tilde{p}\; i\omega_p\left(e^{ip\cdot x}a_p^\dagger - e^{-ip\cdot x}a_p\right),
\end{aligned}$$

where $\Pi(\vec{x})$ is found analogous to $\phi(\vec{x})$ or by $\Pi(\vec{x}) = \partial_t\phi(\vec{x})$.

We used the notation $\phi(\vec{x}), \Pi(\vec{x})$ instead of $\phi(x), \Pi(x)$ in the context of *equal time* commutators. However, of course, the fields also depend on the time. In a more general context we will therefore denote them in terms of the four-vector x instead of $\vec{x}$. This is just a matter of notation and in general it holds $\phi(\vec{x}) = \phi(x), \Pi(\vec{x}) = \Pi(x)$, where in the latter notation the time-dependence is explicit and in the former implicit.

Comment: If we are honest, we didn't really need to include the factors $e^{\pm i\omega_p t}$, when we constructed our $\phi(\vec{p})$ in terms of the ladder operators. However, in the end it worked out quite nicely, as we have now factors of Lorentz invariant exponentials $e^{ip\cdot x}$ in the integral formulas of the out fields $\phi(\vec{x}), \Pi(\vec{x})$. Since ϕ is a scalar field, it is Lorentz invariant and because of the Lorentz invariant exponentials, also the ladder operators are Lorentz invariant. So, as it turned out, we need the factors $e^{\pm i\omega_p t}$, to make the ladder operators Lorentz invariant!

4.5 Energy-Momentum Tensor

4.5.1 Conserved Charges for the Real Klein-Gordon Field

The conserved charges for the real Klein-Gordon field read

$$Q^\nu = \int d^3x\, \mathcal{T}^{0\nu} = \int d^3x\, (\Pi\,(\partial^\nu\phi) - \mathcal{L}\eta^{0\nu}).$$

If we plug in

$$\begin{aligned}
&\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)^2 - \frac{m^2}{2}\phi^2, \\
&\phi(x) = \int d\tilde{p}\;\left(e^{ip\cdot x}a_p^\dagger + e^{-ip\cdot x}a_p\right), \quad \Pi(x) = \int d\tilde{p}\; i\omega_p\left(e^{ip\cdot x}a_p^\dagger - e^{-ip\cdot x}a_p\right),
\end{aligned}$$

we get

$$
\begin{aligned}
Q^\nu = \int d^3x\, d\tilde{p}\, d\tilde{p}' \bigg(& -\omega_p p'^\nu \left(e^{i(p+p')\cdot x} a_p^\dagger a_{p'}^\dagger - e^{-i(p-p')\cdot x} a_p a_{p'}^\dagger - e^{i(p-p')\cdot x} a_p^\dagger a_{p'} \right. \\
& \left. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + e^{-i(p+p')\cdot x} a_p a_{p'} \right) \\
& - \frac{\eta^{0\nu}}{2} \Big(-p^\mu p'_\mu \left(e^{i(p+p')\cdot x} a_p^\dagger a_{p'}^\dagger - e^{-i(p-p')\cdot x} a_p a_{p'}^\dagger - e^{i(p-p')\cdot x} a_p^\dagger a_{p'} + e^{-i(p+p')\cdot x} a_p a_{p'} \right) \\
& - m^2 \big(e^{i(p+p')\cdot x} a_p^\dagger a_{p'}^\dagger + e^{-i(p-p')\cdot x} a_p a_{p'}^\dagger + e^{i(p-p')\cdot x} a_p^\dagger a_{p'} + e^{-i(p+p')\cdot x} a_p a_{p'} \big) \Big) \bigg) \\
= \int d\tilde{p} \frac{1}{2\omega_p} \bigg(& p^\nu \big(e^{i2\omega_p t} a_p^\dagger a_{-p}^\dagger + a_p a_p^\dagger + a_p^\dagger a_p + e^{-i2\omega_p t} a_p a_{-p} \big) \\
& - \frac{\eta^{0\nu}}{2} \Big(p^\mu p_\mu \big(e^{i2\omega_p t} a_p^\dagger a_{-p}^\dagger + a_p a_p^\dagger + a_p^\dagger a_p + e^{-i2\omega_p t} a_p a_{-p} \big) \\
& - m^2 \big(e^{i2\omega_p t} a_p^\dagger a_{-p}^\dagger + a_p a_p^\dagger + a_p^\dagger a_p + e^{-i2\omega_p t} a_p a_{-p} \big) \Big) \bigg).
\end{aligned}
$$

Let's see at only one term, what was done here:

$$
\begin{aligned}
& \int d^3x\, d\tilde{p}' \, \left(-\omega_p p'^\nu \big(e^{i(p+p')\cdot x} a_p^\dagger a_{p'}^\dagger - e^{-i(p-p')\cdot x} a_p a_{p'}^\dagger \big) \right) \\
& \quad = \int d^3x\, d\tilde{p}' \, \left(-\omega_p p'^\nu \big(e^{-i(\vec{p}+\vec{p}')\vec{x}} e^{i(\omega_p+\omega_{p'})\cdot x} a_p^\dagger a_{p'}^\dagger - e^{i(\vec{p}-\vec{p}')\vec{x}} e^{i(\omega_p-\omega_{p'})\cdot x} a_p a_{p'}^\dagger \big) \right) \\
& \quad = \int d\tilde{p}' \, \left(-(2\pi)^3 \omega_p p'^\nu \big(\delta(\vec{p}+\vec{p}') e^{i(\omega_p+\omega_{p'})\cdot x} a_p^\dagger a_{p'}^\dagger - \delta(\vec{p}-\vec{p}') e^{i(\omega_p-\omega_{p'})\cdot x} a_p a_{p'}^\dagger \big) \right) \\
& \quad = -\frac{1}{2} \big(-p^\nu e^{i2\omega_p \cdot x} a_p^\dagger a_{-p}^\dagger - p^\nu a_p a_p^\dagger \big) = \frac{1}{2} p^\nu \big(e^{i2\omega_p \cdot x} a_p^\dagger a_{-p}^\dagger + a_p a_p^\dagger \big),
\end{aligned}
$$

where we used that $p^0 = \omega_p = \omega_{-p}$. Recall also that $d\tilde{p} \coloneqq d^3p/(2\pi)^3 2\omega_p$ and this factor $1/2\omega_p$ must be taken care of when evaluating the δ-function. Now, we also know that

$$
\omega_p = \sqrt{\vec{p}^2 + m^2} \quad \Longrightarrow \quad p^\nu p_\nu = \omega_p^2 - \vec{p} = m^2,
$$

from which we find that the last terms proportional to $\eta^{0\nu}$ cancel and what remains is

$$
Q^\nu = \int d\tilde{p} \frac{1}{2\omega_p} \Big(\omega_p p^\nu \big(a_p a_p^\dagger + a_p^\dagger a_p + e^{i2\omega_p t} a_p^\dagger a_{-p}^\dagger + e^{-i2\omega_p t} a_p a_{-p} \big) \Big).
$$

The latter two terms vanish due to antisymmetry when taking $p \to -p$:

$$
\begin{aligned}
& \int d\tilde{p}\, \omega_p p^\nu e^{i2\omega_p t} a_p^\dagger a_{-p}^\dagger = \frac{1}{2} \int d\tilde{p}\, \big(\omega_p p^\nu e^{i2\omega_p t} a_p^\dagger a_{-p}^\dagger + \omega_p p^\nu e^{i2\omega_p t} a_p^\dagger a_{-p}^\dagger \big) \\
& \quad = \frac{1}{2} \int d\tilde{p}\, \big(\omega_p p^\nu e^{i2\omega_p t} a_p^\dagger a_{-p}^\dagger - \omega_p p^\nu e^{i2\omega_p t} a_{-p}^\dagger a_p^\dagger \big) = 0,
\end{aligned}
$$

since $[a_p^\dagger, a_{p'}^\dagger] = 0$. Thus, we get the conserved charge

$$
Q^\nu = \int d\tilde{p} \frac{1}{2\omega_p} \Big(\omega_p p^\nu \big(a_p a_p^\dagger + a_p^\dagger a_p \big) \Big) = \int d\tilde{p}\, \frac{1}{2\omega_p} \omega_p p^\nu \big(2 a_p^\dagger a_p + [a_p, a_p^\dagger] \big) = \int d\tilde{p}\, p^\nu a_p^\dagger a_p.
$$

where we neglect the constant (no operator) infinity from the commutator $[a_p, a_p^\dagger] = (2\pi)^3 2\omega_p \delta(0)$ as we did before.

4.6 The Fock Space

4.6.1 Commutator of the Four-Momentum Operator and the Creation Operator

We used the following commutator relation:

$$
\begin{aligned}
[P^\mu, a_p^\dagger] &= \int d\tilde{p}\, p'^\mu [a_{p'}^\dagger a_{p'}, a_p^\dagger] = \int d\tilde{p}\, p'^\mu \left(a_{p'}^\dagger [a_{p'}, a_p^\dagger] + \underbrace{[a_{p'}^\dagger, a_p^\dagger]}_{=0} a_{p'} \right) \\
&= \int d\tilde{p}\, (2\pi)^3 2\omega_p \delta(\vec{p}' - \vec{p})\, p'^\mu a_{p'}^\dagger = p^\mu a_p^\dagger.
\end{aligned}
$$

We get the relation for a_p simply by applying a † to this equation:

$$
p^\mu a_p = [P^\mu, a_p^\dagger]^\dagger = [a_p, P^\mu] = -[P^\mu, a_p].
$$

4.6.2 Normalization

$$
\langle p|p'\rangle = \langle 0|a_p a_{p'}^\dagger|0\rangle = \langle 0|[a_p, a_{p'}^\dagger] + a_{p'}^\dagger a_p|0\rangle = (2\pi)^3 2\omega_p \delta(\vec{p} - \vec{p}').
$$

4.8 Causality and Propagators

4.8.1 Deriving $\Delta(z)$

Using

$$
\begin{aligned}
&\phi(\vec{x}) = \int d\tilde{p}\, (e^{ip\cdot x} a_p^\dagger + e^{-ip\cdot x} a_p), \\
&[a_p, a_{p'}] = [a_p^\dagger, a_{p'}^\dagger] = 0, \quad [a_p, a_{p'}^\dagger] = (2\pi)^3 2\omega_p \delta(\vec{p} - \vec{p}'),
\end{aligned}
$$

we find

$$
\begin{aligned}
\Delta(x-y) &:= [\phi(x), \phi(y)] = \int d\tilde{p}\, d\tilde{p}' \left[e^{ip\cdot x} a_p^\dagger + e^{-ip\cdot x} a_p, e^{ip'\cdot y} a_{p'}^\dagger + e^{-ip'\cdot y} a_{p'}\right] \\
&= \int d\tilde{p}\, d\tilde{p}' \left(e^{ip\cdot x} e^{-ip'\cdot y}[a_p^\dagger, a_{p'}] + e^{-ip\cdot x} e^{ip'\cdot y}[a_p, a_{p'}^\dagger]\right) = \int d\tilde{p}\, \left(-e^{ip\cdot(x-y)} + e^{-ip\cdot(x-y)}\right) \\
\Leftrightarrow \quad \Delta(z) &= \int d\tilde{p}\, \left(e^{-ip\cdot z} - e^{ip\cdot z}\right).
\end{aligned}
$$

4.8.2 Deriving $\Delta(z)$ for Complex Fields

Using

$$
\begin{aligned}
&\phi(x) = \int d\tilde{p}\, (b_p^\dagger e^{ip\cdot x} + a_p e^{-ip\cdot x}), \quad \phi^\dagger(x) = \int d\tilde{p}\, (a_p^\dagger e^{ip\cdot x} + b_p e^{-ip\cdot x}), \\
&[a_p, a_{p'}^\dagger] = [b_p, b_{p'}^\dagger] = (2\pi)^3 2\omega_p \delta(\vec{p} - \vec{p}'),
\end{aligned}
$$

with all other commutators vanishing, we find

$$
\begin{aligned}
\Delta(x-y) &:= [\phi(x), \phi^\dagger(y)] = \int d\tilde{p}\, d\tilde{p}' \left[b_p^\dagger e^{ip\cdot x} + a_p e^{-ip\cdot x}, a_{p'}^\dagger e^{ip'\cdot y} + b_{p'} e^{-ip'\cdot y}\right] \\
&= \int d\tilde{p}\, d\tilde{p}' \left(e^{ip\cdot x} e^{-ip'\cdot y}[b_p^\dagger, b_{p'}] + e^{-ip\cdot x} e^{ip'\cdot y}[a_p, a_{p'}^\dagger]\right) = \int d\tilde{p}\, \left(-e^{ip\cdot(x-y)} + e^{-ip\cdot(x-y)}\right) \\
\Leftrightarrow \quad \Delta(z) &= \int d\tilde{p}\, \left(e^{-ip\cdot z} - e^{ip\cdot z}\right).
\end{aligned}
$$

4.8.3 Vacuum Expectation Value of $\boldsymbol{\phi(x)\phi^\dagger(y)}$

Using the complex Klein-Gordon fields in terms of the latter operators, as well as $a_p|0\rangle = 0$ yields

$$
\begin{aligned}
\langle 0|\phi(x)\phi^\dagger(y)|0\rangle &= \int d\tilde{p}\, d\tilde{p}'\, e^{-ip\cdot x}e^{ip'\cdot y}\langle 0|a_p a_{p'}^\dagger|0\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\, e^{-ip\cdot x}e^{ip'\cdot y}\langle 0|a_{p'}^\dagger a_p + [a_p, a_{p'}^\dagger]|0\rangle = \int d\tilde{p}\, d\tilde{p}'\, e^{-ip\cdot x}e^{ip'\cdot y}\,(2\pi)^3 2\omega_p \delta(\vec{p}-\vec{p}') \\
&= \int d\tilde{p}\, e^{-ip\cdot(x-y)}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\langle 0|\phi^\dagger(y)\phi(x)|0\rangle &= \int d\tilde{p}\, d\tilde{p}'\, e^{-ip\cdot y}e^{ip'\cdot x}\langle 0|b_p b_{p'}^\dagger|0\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\, e^{-ip\cdot y}e^{ip'\cdot x}\langle 0|b_{p'}^\dagger b_p + [b_p, b_{p'}^\dagger]|0\rangle = \int d\tilde{p}\, e^{-ip\cdot(y-x)} = \int d\tilde{p}\, e^{ip\cdot(x-y)}.
\end{aligned}
$$

4.8.4 The Feynman Propagator

We defined the Feynman propagator

$$
D_F(x-y) = \begin{cases} \langle 0|\phi(x)\phi^\dagger(y)|0\rangle, & x^0 > y^0 \\ \langle 0|\phi^\dagger(y)\phi(x)|0\rangle, & y^0 > x^0 \end{cases} = \begin{cases} \int d\tilde{p}\, e^{-ip\cdot(x-y)}, & x^0 > y^0 \\ \int d\tilde{p}\, e^{ip\cdot(x-y)}, & y^0 > x^0 \end{cases}.
$$

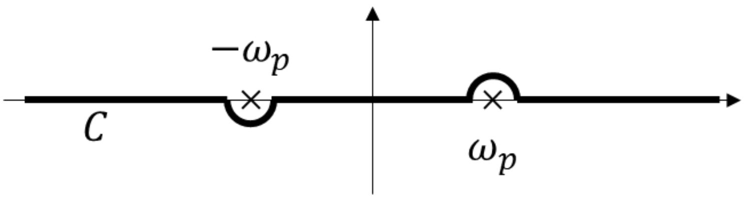
This Feynman propagator can be written as (recall $d^n\bar{p} \coloneqq d^n p/(2\pi)^n$)

$$
D_F(x-y) = \int d^4\bar{p}\, \frac{i}{p^2-m^2} e^{-ip\cdot(x-y)}.
$$

Let's see why (we state the result in the beginning, since it is easier to do the proof backwards). Note that this is the first time, we integrate over the four-momentum, not the three-momentum with p_0 fixed by $p_0 = \sqrt{\vec{p}^2+m^2} =: \omega_p$. Thus, we *do not assume* this energy momentum relation here. Consider

$$
\frac{1}{p^2-m^2} = \frac{1}{p_0^2-\vec{p}^2-m^2} = \frac{1}{p_0^2-\omega_p^2} = \frac{1}{(p_0+\omega_p)(p_0-\omega_p)}.
$$

Thus, our integral has poles at $p_0 = \pm\omega_p$. We can avoid those poles, if we integrate p_0 along a contour C in the complex plain: We will depart the real axis by an infinitesimal amount at the poles. If we do it in the following way, we can construct the different cases $x^0 \lessgtr y^0$ of the definition of the Feynman propagator:



Let's consider $x^0 > y^0$ and try to close the contour such that we can apply the residues theorem.[1] For $x^0 > y^0$ the exponential in our integral reads

$$
e^{-ip_0\cdot(x^0-y^0)}.
$$

[1] The residue's theorem states that

$$
\oint_C dz\, f(z) = 2\pi i \sum_i \operatorname{res}(f, z_i), \quad \operatorname{res}(f, z_i) = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_i)^n f(z)\Big|_{z=z_i},
$$

where the z_i are all the poles inside C and we integrate counter-clockwise. n is the order of the pole. Integrating clockwise gives an overall minus sign.

Therefore, we might want to close the contour in the lower half plane, because if we sent $p_0 \to -i\infty$, this exponential will give us zero and the contour within the complex plain doesn't contribute. Using the residue's theorem to perform the p^0 integral, our total integral then reads (note that we get a minus sign from integrating clockwise)

$$D_F(x-y) = \int d^4\bar{p} \frac{i}{(p_0+\omega_p)(p_0-\omega_p)} e^{-ip\cdot(x-y)} = -2\pi i \int \frac{d^3p}{(2\pi)^4} \frac{i}{p_0+\omega_p} e^{-ip\cdot(x-y)}\bigg|_{p_0=\omega_p}$$
$$= \int d\tilde{p}\, e^{-ip\cdot(x-y)}\bigg|_{p_0=\omega_p}.$$

This is indeed what we defined D_F to be for $x^0 > y^0$ (integrals with a measure $d\tilde{p}$ obey the condition $p_0 = \omega_p$ per construction; it is redundant to explicitly write it down).

In the case of $y^0 > x^0$ we need to close the contour in the upper half plane to make the contribution of this part of the contour vanish due to the exponential (it vanishes now for $p_0 \to i\infty$). Now we get

$$D_F(x-y) = \int d^4\bar{p} \frac{i}{(p_0+\omega_p)(p_0-\omega_p)} e^{-ip\cdot(x-y)} = 2\pi i \int d^4\bar{p} \frac{i}{p_0-\omega_p} e^{-ip\cdot(x-y)}\bigg|_{p_0=-\omega_p}$$
$$= \int d^3\tilde{p}\, e^{-ip\cdot(x-y)}\bigg|_{p_0=-\omega_p} = \int d^3\tilde{p}\, e^{ip\cdot(x-y)}\bigg|_{p_0=\omega_p}.$$

The last step contains the following modification:

$$e^{-ip\cdot(x-y)}\big|_{p_0=-\omega_p} = e^{-ip_0\cdot(x^0-y^0)} e^{-i\vec{p}(\vec{x}-\vec{y})}\big|_{p_0=-\omega_p} = e^{ip_0\cdot(x^0-y^0)} e^{-i\vec{p}(\vec{x}-\vec{y})}\big|_{p_0=\omega_p},$$

and finally, we rotated $\vec{p} \to -\vec{p}$ to get the desired result. And this is what we defined D_F to be for $y^0 > x^0$.

So, the integral

$$\int d^4\bar{p} \frac{i}{p^2-m^2} e^{-ip\cdot(x-y)}$$

gives the Feynman propagator *when we integrate* p_0 *along* C (where C is only the contour along the real axes with the small detours but does not include the large half circle far at $p_0 \to \pm i\infty$ in the complex plane). It is now more elegant to define the integral not as "when integrated along C" but to integrate simply along the real axes and shift the poles an infinitesimal amount into the complex plain. Of course, we must watch out, to shift it into the right direction, to be equivalent to "integrate along C":

$$\frac{1}{(p_0+\omega_p)(p_0-\omega_p)} \to \frac{1}{\big(p_0-(-\omega_p+i\epsilon)\big)\big(p_0-(\omega_p-i\epsilon)\big)}.$$

We now have one pole at $p_0 = -\omega_p + i\epsilon$ and $p_0 = \omega_p - i\epsilon$ as desired. Finally, we want to go back from p_0, ω_p to p, m in the denominator:

$$\frac{1}{(p_0+\omega_p-i\epsilon)(p_0-\omega_p+i\epsilon)} = \frac{1}{\big(p_0+(\omega_p-i\epsilon)\big)\big(p_0-(\omega_p-i\epsilon)\big)} = \frac{1}{p_0^2-(\omega_p-i\epsilon)^2}$$
$$= \frac{1}{p_0^2-\omega_p^2+2\omega_p i\epsilon+\mathcal{O}(\epsilon^2)} \mathrel{\widehat{=}} \frac{1}{p_0^2-\omega_p^2+i\epsilon} = \frac{1}{p^2-m^2+i\epsilon}$$

where we changed $\epsilon \to \epsilon/2\omega_p$, which is still infinitesimal. In this language, we can write the Feynman propagator as

$$D_F(x-y) = \int d^4\bar{p} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}.$$

5 QUANTIZED DIRAC FIELD

5.1 Quantization of the Dirac Field

5.1.1 Expression for the Ladder Operators

We can give the ladder operators as

$$a_{\alpha p} = e^{i\omega_p t}\bar{u}_{\alpha p}\gamma^0\psi^-(\vec{p}), \quad b^\dagger_{\alpha p} = e^{-i\omega_p t}\bar{v}_{\alpha p}\gamma^0\psi^+(\vec{p}),$$
$$\psi^\pm(\vec{p}) \coloneqq \int d^3x\, e^{\pm i\vec{p}\vec{x}}\psi(\vec{x}).$$

For proving this, we will plug our $\psi(x)$ into those expressions:

$$\begin{aligned}
a_{\alpha p} &= e^{i\omega_p t}\bar{u}_{\alpha p}\gamma^0\psi^-(\vec{p}) = \int d^3x\, e^{ip\cdot x}\bar{u}_{\alpha p}\gamma^0\psi(\vec{x}) \\
&= \int d^3x\, d\tilde{p}'\, \bar{u}_{\alpha p}\gamma^0\left(b^\dagger_{\sigma p'}v^\sigma_{p'}e^{i(p'+p)\cdot x} + a_{\sigma p'}u^\sigma_{p'}e^{-i(p'-p)\cdot x}\right) \\
&= \frac{1}{2\omega_p}\bar{u}_{\alpha p}\gamma^0\left(b^\dagger_{\sigma,-p}v^\sigma_{-p}e^{i2\omega_p t} + a_{\sigma p}u^\sigma_p\right).
\end{aligned}$$

In the last step, the d^3x integral was evaluated to turn the exponential into δ-functions, which then where used to evaluate the $d\tilde{p}'$ integral. Now, we use the normalization condition of Dirac spinors $u^\dagger u = 2\omega_p$,

$$\bar{u}_{\alpha p}\gamma^0 u^\sigma_p = u^\dagger_{\alpha p}u^\sigma_p = 2\omega_p\delta^\sigma_\alpha,$$

as well as

$$2\omega_p\bar{u}_{\alpha p}\gamma^0 v^\sigma_{-p} = \bar{u}_{\alpha p}\left(\omega_p\gamma^0 - \vec{\gamma}\vec{p} + \vec{\gamma}\vec{p} + \omega_p\gamma^0\right)v^\sigma_{-p}.$$

The first two terms in the bracket yield, using $\bar{u}_{\alpha p}(\not{p} - m) = 0$,

$$\bar{u}_{\alpha p}\left(\omega_p\gamma^0 - \vec{\gamma}\vec{p}\right) = \bar{u}_{\alpha p}\not{p} = \bar{u}_{\alpha p}m.$$

The last two terms yield, using $(\not{p} + m)v_p = 0$,

$$\left(\vec{\gamma}\vec{p} + \omega_p\gamma^0\right)v^\sigma_{-p} = \left(-\vec{\gamma}\vec{p} + \omega_p\gamma^0\right)v^\sigma_p = \left(-\vec{\gamma}\vec{p} + \omega_p\gamma^0\right)v^\sigma_p = \not{p}v^\sigma_p = -mv^\sigma_p.$$

Thus, we get

$$2\omega_p\bar{u}_{\alpha p}\gamma^0 v^\sigma_{-p} = \bar{u}_{\alpha p}(m - m)v^\sigma_{-p} = 0 \quad \Leftrightarrow \quad \bar{u}_{\alpha p}\gamma^0 v^\sigma_{-p} = 0.$$

Using those relations, we find

$$a_{\alpha p} = \frac{1}{2\omega_p}\left(b^\dagger_{\sigma,-p}\underbrace{\bar{u}_{\alpha p}\gamma^0 v^\sigma_{-p}}_{=0}e^{i2\omega_p t} + a_{\sigma p}\underbrace{\bar{u}_{\alpha p}\gamma^0 u^\sigma_p}_{=2\omega_p\delta^\sigma_\alpha}\right) = a_{\alpha p},$$

what was to be proven. In the same way, we find

$$\begin{aligned}
b^\dagger_{\alpha p} &= e^{-i\omega_p t}\bar{v}_{\alpha p}\gamma^0\psi^+(\vec{p}) = \int d^3x\, e^{-ip\cdot x}\bar{v}_{\alpha p}\gamma^0\psi(\vec{x}) \\
&= \int d^3x\, d\tilde{p}\, \bar{v}_{\alpha p}\gamma^0\left(b^\dagger_{\sigma p'}v^\sigma_{p'}e^{i(p'-p)\cdot x} + a_{\sigma p'}u^\sigma_{p'}e^{-i(p'+p)\cdot x}\right) \\
&= \frac{1}{2\omega_p}\bar{v}_{\alpha p}\gamma^0\left(b^\dagger_{\sigma p}v^\sigma_p + a_{\sigma,-p}u^\sigma_{-p}e^{-i2\omega_p t}\right) = \frac{1}{2\omega_p}b^\dagger_{\sigma p}v^\dagger_{\alpha p}v^\sigma_p = b^\dagger_{\alpha p}.
\end{aligned}$$

5.2 The Four-Momentum Operator

5.2.1 The Four-Momentum Operator of the Dirac Field

This is the analogous calculation to the one in (>4.5.1). Using the Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = 0$ we find the energy momentum tensor

$$\mathcal{T}^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial^\nu\psi + \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}}_{=0}\partial^\nu\bar{\psi} - \underbrace{\mathcal{L}}_{=0}\eta^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi.$$

Thus, the conserved charge, i.e. the four-momentum operator, reads

$$\begin{aligned}
P^\nu &= \int d^3x\, \mathcal{T}^{0\nu} = \int d^3x\, i\psi^\dagger\partial^\nu\psi \\
&= \int d^3x\, d\tilde{p}\, d\tilde{p}'\, (-p'^\nu)\big(a^\dagger_{\alpha p}u^{\dagger\alpha}_p e^{ip\cdot x} + b_{\alpha p}v^{\dagger\alpha}_p e^{-ip\cdot x}\big)\big(b^\dagger_{\sigma p'}v^\sigma_{p'}e^{ip'\cdot x} - a_{\sigma p'}u^\sigma_{p'}e^{-ip'\cdot x}\big) \\
&= \int d\tilde{p}\, d\tilde{p}'\, (2\pi)^3(-p'^\nu)\big(\delta(\vec{p}+\vec{p}')e^{i(\omega_p+\omega_{p'})t}a^\dagger_{\alpha p}b^\dagger_{\sigma p'}u^{\dagger\alpha}_p v^\sigma_{p'} \\
&\quad - \delta(\vec{p}-\vec{p}')e^{i(\omega_p-\omega_{p'})t}a^\dagger_{\alpha p}a_{\sigma p'}u^{\dagger\alpha}_p u^\sigma_{p'} + \delta(\vec{p}-\vec{p}')e^{-i(\omega_p-\omega_{p'})t}b_{\alpha p}b^\dagger_{\sigma p'}v^{\dagger\alpha}_p v^\sigma_{p'} \\
&\quad - \delta(\vec{p}+\vec{p}')e^{-i(\omega_p+\omega_{p'})t}b_{\alpha p}a_{\sigma p'}v^{\dagger\alpha}_p u^\sigma_{p'}\big) \\
&= \int d\tilde{p}\,\frac{1}{2\omega_p}(-p^\nu)\big(-e^{i2\omega_p t}a^\dagger_{\alpha p}b^\dagger_{\sigma,-p}u^{\dagger\alpha}_p v^\sigma_{-p} - a^\dagger_{\alpha p}a_{\sigma p}u^{\dagger\alpha}_p u^\sigma_p + b_{\alpha p}b^\dagger_{\sigma p}v^{\dagger\alpha}_p v^\sigma_p \\
&\quad + e^{-i2\omega_p t}b_{\alpha p}a_{\sigma,-p}v^{\dagger\alpha}_p u^\sigma_{-p}\big)
\end{aligned}$$

We already saw in (>5.1.1) that

$$\bar{u}_{\alpha p}\gamma^0 u^\sigma_p = u^\dagger_{\alpha p}u^\sigma_p = 2\omega_p\delta^\sigma_\alpha, \quad \bar{u}_{\alpha p}\gamma^0 v^\sigma_{-p} = u^\dagger_{\alpha p}v^\sigma_{-p} = 0.$$

Thus, we find

$$P^\nu = \int d\tilde{p}\, p^\nu\big(a^\dagger_{\alpha p}a_{\alpha p} - b_{\alpha p}b^\dagger_{\alpha p}\big).$$

5.2.2 The Four-Momentum Operator using Anticommutator Relations

Using

$$\{b_{\alpha p}, b^\dagger_{\sigma p'}\} = b_{\alpha p}b^\dagger_{\sigma p'} + b^\dagger_{\sigma p'}b_{\alpha p} = (2\pi)^3 2\omega_p\delta_{\alpha\sigma}\delta(\vec{p}-\vec{p}'),$$

we can give the four-momentum operator as

$$\begin{aligned}
P^\nu &= \int d\tilde{p}\, p^\nu\big(a^\dagger_{\alpha p}a_{\alpha p} - b_{\alpha p}b^\dagger_{\alpha p}\big) = \int d\tilde{p}\, p^\nu\left(a^\dagger_{\alpha p}a_{\alpha p} - \big(-b^\dagger_{\alpha p}b_{\alpha p} + \{b_{\alpha p}, b^\dagger_{\sigma p'}\}\big)\right) \\
&= \int d\tilde{p}\, p^\nu\big(a^\dagger_{\alpha p}a_{\alpha p} + b^\dagger_{\alpha p}b_{\alpha p}\big),
\end{aligned}$$

where we again neglect the infinite constant.

5.3 Anticommutator Relations

5.3.1 Anticommutator Relations of the Ladder Operators

We assume

$$\{\psi(x), \psi^\dagger(y)\} = \delta(\vec{x}-\vec{y})$$

and want to show that we do actually get this result if we use

$$\{a_{\alpha p}, a^\dagger_{\sigma p'}\} = \{b_{\alpha p}, b^\dagger_{\sigma p'}\} = (2\pi)^3 2\omega_p\delta_{\alpha\sigma}\delta(\vec{p}-\vec{p}')$$

and all other anticommutators zero. We will also need the completeness relation of the spinors,

$$u_p^\alpha \bar{u}_p^\alpha = \not{p} + m, \quad v_p^\alpha \bar{v}_p^\alpha = \not{p} - m.$$

If we recall our expansions in terms of the ladder operators we get

$$\begin{aligned}
\{\psi(x), \psi^\dagger(y)\} &= \int d\tilde{p}\, d\tilde{p}' \left\{b_{\alpha p}^\dagger v_p^\alpha e^{ip\cdot x} + a_{\alpha p} u_p^\alpha e^{-ip\cdot x}, a_{\sigma p'}^\dagger u_{p'}^{\dagger\sigma} e^{ip'\cdot y} + b_{\sigma p'} v_{p'}^{\dagger\sigma} e^{-ip'\cdot y}\right\} \\
&= \int d\tilde{p}\, d\tilde{p}' \left(e^{ip\cdot x} e^{ip'\cdot y} v_p^\alpha u_{p'}^{\dagger\sigma}\{b_{\alpha p}^\dagger, a_{\sigma p'}^\dagger\} + e^{ip\cdot x} e^{-ip'\cdot y} v_p^\alpha v_{p'}^{\dagger\sigma}\{b_{\alpha p}^\dagger, b_{\sigma p'}\}\right. \\
&\left.+ e^{-ip\cdot x} e^{ip'\cdot y} u_p^\alpha u_{p'}^{\dagger\sigma}\{a_{\alpha p}, a_{\sigma p'}^\dagger\} + e^{-ip\cdot x} e^{-ip'\cdot y} u_p^\alpha v_{p'}^{\dagger\sigma}\{a_{\alpha p}, b_{\sigma p'}\}\right) \\
&= \int d\tilde{p} \left(e^{ip\cdot(x-y)} v_p^\alpha \bar{v}_p^\alpha \gamma^0 + e^{-ip\cdot(x-y)} u_p^\alpha \bar{u}_p^\alpha \gamma^0\right) \\
&= \int d\tilde{p} \left(e^{ip\cdot(x-y)} (\not{p} - m)\gamma^0 + e^{-ip\cdot(x-y)} (\not{p} + m)\gamma^0\right)
\end{aligned}$$

If we now rotate $\vec{p} \to -\vec{p}$ in the second term of the sum in the brackets, all we're left with is

$$\{\psi(x), \psi^\dagger(y)\} = \int d\tilde{p} \left(e^{ip\cdot(x-y)} \omega_p + e^{ip\cdot(x-y)} \omega_p\right) = \delta(x - y).$$

There seems to be a time-component in the exponent, which is not rotated but still catches a minus sign. But recall that those anticommutators are equal time anticommutators, so actually the time-component in the exponent is zero.

5.4 The Fock Space

5.4.1 Commutator of the Four-Momentum Operator and the Creation Operator

Using the identity

$$[AB, C] = A\{B, C\} - \{A, C\}B$$

yields

$$\begin{aligned}
\left[P^\mu, a_{\sigma p}^\dagger\right] &= \int d\tilde{p}'\, p'^\mu \left[a_{\alpha p'}^\dagger a_{\alpha p'} + b_{\alpha p'}^\dagger b_{\alpha p'}, a_{\sigma p}^\dagger\right] \\
&= \int d\tilde{p}'\, p'^\mu \left(a_{\alpha p'}^\dagger \{a_{\alpha p'}, a_{\sigma p}^\dagger\} - \{a_{\alpha p'}^\dagger, a_{\sigma p}^\dagger\} a_{\alpha p'} + b_{\alpha p'}^\dagger \{b_{\alpha p'}, a_{\sigma p}^\dagger\} - \{b_{\alpha p'}^\dagger, a_{\sigma p}^\dagger\} b_{\alpha p'}\right) \\
&= \int d\tilde{p}'\, p'^\mu\, a_{\alpha p'}^\dagger\, (2\pi)^3 2\omega_p \delta_{\alpha\sigma} \delta(\vec{p} - \vec{p}') = p^\mu\, a_{\sigma p}^\dagger.
\end{aligned}$$

We get the relation for a_p simply by applying a † to this equation:

$$p^\mu a_{\alpha p} = \left[P^\mu, a_{\alpha p}^\dagger\right]^\dagger = \left[a_{\alpha p}, P^\mu\right] = -\left[P^\mu, a_{\alpha p}\right].$$

$b_{\alpha p}$ has the same commutator relations:

$$\begin{aligned}
\left[P^\mu, b_{\sigma p}^\dagger\right] &= \int d\tilde{p}'\, p'^\mu \left[a_{\alpha p'}^\dagger a_{\alpha p'} + b_{\alpha p'}^\dagger b_{\alpha p'}, b_{\sigma p}^\dagger\right] \\
&= \int d\tilde{p}'\, p'^\mu \left(a_{\alpha p'}^\dagger \{a_{\alpha p'}, b_{\sigma p}^\dagger\} - \{a_{\alpha p'}^\dagger, b_{\sigma p}^\dagger\} a_{\alpha p'} + b_{\alpha p'}^\dagger \{b_{\alpha p'}, b_{\sigma p}^\dagger\} - \{b_{\alpha p'}^\dagger, b_{\sigma p}^\dagger\} b_{\alpha p'}\right) \\
&= \int d\tilde{p}'\, p'^\mu\, b_{\alpha p'}^\dagger\, (2\pi)^3 2\omega_p \delta_{\alpha\sigma} \delta(\vec{p} - \vec{p}') = p^\mu\, b_{\sigma p}^\dagger.
\end{aligned}$$

5.4.2 Normalization

$$\langle \alpha, p | \sigma, p' \rangle = \left\langle 0 \middle| a_{\alpha p} a_{\sigma p'}^\dagger \middle| 0 \right\rangle = \left\langle 0 \middle| -a_{\sigma p'}^\dagger a_{\alpha p} + \{a_{\alpha p}, a_{\sigma p'}^\dagger\} \middle| 0 \right\rangle = (2\pi)^3 2\omega_p \delta_{\alpha\sigma} \delta(\vec{p} - \vec{p}').$$

5.5 Causality and Propagators

5.5.1 Deriving $\widetilde{\boldsymbol{\Delta}}(\boldsymbol{z})$

Using the field expansions in the ladder operators as well as their commutator relations and the completeness of the spinors,

$$u_p^\alpha \bar{u}_p^\alpha = \not{p} + m, \quad v_p^\alpha \bar{v}_p^\alpha = \not{p} - m,$$

yields

$$\begin{aligned}
\widetilde{\Delta}(x-y) &\coloneqq \{\psi(x), \bar{\psi}(y)\} \\
&= \int d\tilde{p}\, d\tilde{p}'\, \{b_{\alpha p}^\dagger v_p^\alpha e^{ip\cdot x} + a_{\alpha p} u_p^\alpha e^{-ip\cdot x}, a_{\sigma p'}^\dagger \bar{u}_{p'}^\sigma e^{ip'\cdot y} + b_{\sigma p'} \bar{v}_{p'}^\sigma e^{-ip'\cdot y}\} \\
&= \int d\tilde{p}\, d\tilde{p}'\, \left(e^{ip\cdot x} e^{-ip'\cdot y} v_p^\alpha \bar{v}_{p'}^\sigma \{b_{\alpha p}^\dagger, b_{\sigma p'}\} + e^{-ip\cdot x} e^{ip'\cdot y} u_p^\alpha \bar{u}_{p'}^\sigma \{a_{\alpha p}, a_{\sigma p'}^\dagger\}\right) \\
&= \int d\tilde{p}\, \left(e^{ip\cdot(x-y)} v_p^\alpha \bar{v}_p^\alpha + e^{-ip\cdot(x-y)} u_p^\alpha \bar{u}_p^\alpha\right) = \int d\tilde{p}\, \left(e^{ip\cdot(x-y)}(\not{p} - m) + e^{-ip\cdot(x-y)}(\not{p} + m)\right) \\
&= (i\not{\partial}_x + m) \int d\tilde{p}\, \left(-e^{ip\cdot(x-y)} + e^{-ip\cdot(x-y)}\right) = (i\not{\partial}_x + m)\Delta(x-y)
\end{aligned}$$

$$\Leftrightarrow \quad \widetilde{\Delta}(z) = (i\not{\partial}_z + m) \int d\tilde{p}\, \left(-e^{ip\cdot z} + e^{-ip\cdot z}\right) = (i\not{\partial}_z + m)\Delta(z).$$

5.5.2 The Feynman Propagator

Using the field expansions in the ladder operators as well as their commutator relations and the completeness of the spinors,

$$u_p^\alpha \bar{u}_p^\alpha = \not{p} + m, \quad v_p^\alpha \bar{v}_p^\alpha = \not{p} - m,$$

yields

$$\begin{aligned}
\langle 0|\psi(x)\bar{\psi}(y)|0\rangle &= \int d\tilde{p}\, d\tilde{p}'\, \langle 0|\left(b_{\alpha p}^\dagger v_p^\alpha e^{ip\cdot x} + a_{\alpha p} u_p^\alpha e^{-ip\cdot x}\right)\left(a_{\sigma p'}^\dagger \bar{u}_{p'}^\sigma e^{ip'\cdot y} + b_{\sigma p'} \bar{v}_{p'}^\sigma e^{-ip'\cdot y}\right)|0\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\, e^{-ip\cdot x} e^{ip'\cdot y} u_p^\alpha \bar{u}_{p'}^\sigma \langle 0|a_{\alpha p} a_{\sigma p'}^\dagger|0\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\, e^{-ip\cdot x} e^{ip'\cdot y} u_p^\alpha \bar{u}_{p'}^\sigma \langle 0|-a_{\sigma p'}^\dagger a_{\alpha p} + \{a_{\alpha p}, a_{\sigma p'}^\dagger\}|0\rangle = \int d\tilde{p}\, e^{-ip\cdot(x-y)} u_p^\alpha \bar{u}_p^\alpha \\
&= \int d\tilde{p}\, e^{-ip\cdot(x-y)}(\not{p} + m) = (i\not{\partial}_x + m) \int d\tilde{p}\, e^{-ip\cdot(x-y)} = (i\not{\partial}_x + m)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle
\end{aligned}$$

as well as

$$\begin{aligned}
\langle 0|\bar{\psi}(y)\psi(x)|0\rangle &= \int d\tilde{p}\, d\tilde{p}'\, \langle 0|\left(a_{\alpha p}^\dagger \bar{u}_p^\alpha e^{ip\cdot y} + b_{\alpha p} \bar{v}_p^\alpha e^{-ip\cdot y}\right)\left(b_{\sigma p'}^\dagger v_{p'}^\sigma e^{ip'\cdot x} + a_{\sigma p'} u_{p'}^\sigma e^{-ip'\cdot x}\right)|0\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\, e^{-ip\cdot y} e^{ip'\cdot x} \bar{v}_p^\alpha v_{p'}^\sigma \langle 0|b_{\alpha p} b_{\sigma p'}^\dagger|0\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\, e^{-ip\cdot y} e^{ip'\cdot x} \bar{v}_p^\alpha v_{p'}^\sigma \langle 0|-b_{\sigma p'}^\dagger b_{\alpha p} + \{b_{\alpha p}, b_{\sigma p'}^\dagger\}|0\rangle = \int d\tilde{p}\, e^{ip\cdot(x-y)}(\not{p} - m) \\
&= (-i\not{\partial}_x - m) \int d\tilde{p}\, e^{ip\cdot(x-y)} = -(i\not{\partial}_x + m)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle.
\end{aligned}$$

The last equal sign makes use of (>4.8.3). Thus, we find that we can write

$$\begin{aligned}
\widetilde{D}_F(x-y) &= \begin{cases} +\langle 0|\psi(x)\bar{\psi}(y)|0\rangle, & x^0 \geq y^0 \\ -\langle 0|\bar{\psi}(y)\psi(x)|0\rangle, & y^0 \geq x^0 \end{cases} = (i\partial_x + m) \begin{cases} \langle 0|\phi(x)\phi^\dagger(y)|0\rangle, & x^0 \geq y^0 \\ \langle 0|\phi^\dagger(y)\phi(x)|0\rangle, & y^0 \geq x^0 \end{cases} \\
&= (i\partial_x + m) D_F(x-y).
\end{aligned}$$

5.5.3 Elegant Form of the Feynman Propagator

Since $p^\mu p^\nu$ commute, they are symmetric in μ, ν and we get the identity

$$\not{p}^2 = \gamma_\mu \gamma_\nu p^\mu p^\nu = \frac{1}{2}\left(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu\right) p^\mu p^\nu = \frac{1}{2}\{\gamma_\mu, \gamma_\nu\} p^\mu p^\nu = g_{\mu\nu} p^\mu p^\nu$$

with which we can write[1]

$$\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} = \frac{\not{p} + m}{p^2 - (m^2 - i\epsilon)} = \frac{\not{p} + m}{p^2 - (m - i\tilde{\epsilon})^2} = \frac{\not{p} + m}{\left(\not{p} - (m - i\tilde{\epsilon})\right)\left(\not{p} + (m - i\tilde{\epsilon})\right)}$$
$$= \frac{1}{\not{p} - m + i\tilde{\epsilon}}.$$

Since $\tilde{\epsilon}$ is still infinitesimal, we can of course write again ϵ instead of $\tilde{\epsilon}$.

[1] This appears to be scary: We have a matrix in the denominator! But we simply understand this object as we derived it. If it makes us feel better, we could *define* this object as

$$\frac{1}{\not{p} - m + i\tilde{\epsilon}} \coloneqq \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}.$$

6 QUANTIZED EM FIELD

6.1 Gauge Fixing

6.1.1 Conjugate Momentum of the Electromagnetic Field

Using the Lagrangian $\mathcal{L} = -F_{\mu\nu}F^{\mu\nu}$, the conjugate momentum reads

$$\begin{aligned}
\Pi_\nu &= \frac{\partial\mathcal{L}}{\partial\dot{A}^\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^0 A^\nu)} = -\frac{1}{2}(\partial_\sigma A_\kappa - \partial_\kappa A_\sigma)\frac{\partial}{\partial(\partial^0 A^\nu)}(\partial^\sigma A^\kappa - \partial^\kappa A^\sigma) \\
&= -\frac{1}{2}(\partial_\sigma A_\kappa - \partial_\kappa A_\sigma)(\delta_0^\sigma \delta_\nu^\kappa - \delta_0^\kappa \delta_\nu^\sigma) = -\frac{1}{2}\big((\partial_0 A_\nu - \partial_\nu A_0) - (\partial_\nu A_0 - \partial_0 A_\nu)\big) \\
&= -(\partial_0 A_\nu - \partial_\nu A_0) = -F_{0\nu}.
\end{aligned}$$

6.1.2 Euler-Lagrange/Maxwell equations in Lorentz Gauge

In section 3.6 we saw that the Euler-Lagrange/Maxwell equations in general read

$$\partial_\nu F^{\nu\mu} = j^\mu.$$

If we impose the Lorentz gauge $\partial_\mu A^\mu = 0$, we find

$$j^\mu = \partial_\nu F^{\nu\mu} = \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = \partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu = \partial_\nu \partial^\nu A^\mu.$$

6.1.3 Euler-Lagrange/Maxwell equations with a modified Lagrangian

If we consider the modified EM-field Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 - j_\mu A^\mu,$$

without imposing the Lorentz gauge, we still get the same equations of motion. When evaluating the Euler-Lagrange equations, note that we can reuse our result from (>3.6.1) and just add the contribution of the additional term:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial A^\mu} - \partial^\nu\frac{\partial\mathcal{L}}{\partial(\partial^\nu A^\mu)} &= \underbrace{-j_\mu + \partial^\nu F_{\nu\mu}}_{\text{from (>3.6.1)}} - \partial^\nu\left(\frac{\partial}{\partial(\partial^\nu A^\mu)}\left(-\frac{\lambda}{2}(\partial_\sigma A^\sigma)^2\right)\right) \\
&= -j_\mu + \partial^\nu F_{\nu\mu} + \frac{\lambda}{2}\partial^\nu\left(\frac{\partial}{\partial(\partial^\nu A^\mu)}(\partial_\sigma A^\sigma)(\partial_\kappa A^\kappa)\right) \\
&= -j_\mu + \partial^\nu F_{\nu\mu} + \lambda\partial^\nu\left((\partial_\sigma A^\sigma)\frac{\partial}{\partial(\partial^\nu A^\mu)}\left(\eta_{\kappa\eta}\partial^\eta A^\kappa\right)\right) = -j_\mu + \partial^\nu F_{\nu\mu} + \lambda\eta_{\mu\nu}\partial^\nu\partial_\sigma A^\sigma \\
&= -j_\mu + \partial^\nu\left(\partial_\nu A_\mu - \partial_\mu A_\nu\right) + \lambda\partial_\mu\partial_\sigma A^\sigma = -j_\mu + \partial^\nu\partial_\nu A_\mu + (\lambda - 1)\partial_\mu\partial^\nu A_\nu \overset{!}{=} 0
\end{aligned}$$

λ was introduced as some arbitrary parameter. Different choices of λ are also referred to as different "gauges", although this has nothing to do with, for example, Lorentz or Coulomb gauge. If we choose the Feynman "gauge" $\lambda = 1$, we find

$$\partial_\nu \partial^\nu A^\mu = j^\mu.$$

6.1.4 Canonical Momentum with Modified Lagrangian

We want to calculate the canonical momentum of the modified Lagrangian from (>6.1.3). For this, we can reuse the canonical momentum of the original Lagrangian from (>6.1.1) and simply add the contribution of the additional term:

$$\begin{aligned}
\Pi_\nu &= \frac{\partial\mathcal{L}}{\partial\dot{A}^\nu} = -F_{0\nu} - \frac{\lambda}{2}\frac{\partial}{\partial(\partial^0 A^\nu)}\left(\partial_\mu A^\mu\right)(\partial_\sigma A^\sigma) = -F_{0\nu} - \lambda\left(\partial_\mu A^\mu\right)\frac{\partial(\partial_\sigma A^\sigma)}{\partial(\partial^0 A^\nu)} \\
&= -F_{0\nu} - \lambda\left(\partial_\mu A^\mu\right)\delta_{\sigma 0}\delta_\nu^\sigma = -F_{0\nu} - \lambda\left(\partial_\mu A^\mu\right)\delta_{\nu 0}.
\end{aligned}$$

In the Feynman „gauge“ $\lambda = 0$ this yields

$$\Pi^0 = -\partial_\mu A^\mu, \quad \Pi^i = -F^{0i}.$$

6.1.5 Even Simpler Lagrangian

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = -j_\mu - \partial^\nu \frac{\partial}{\partial(\partial^\nu A^\mu)}\left(-\frac{1}{2}(\partial_\sigma A_\kappa)(\partial^\sigma A^\kappa)\right) = -j_\mu + \partial^\nu(\partial_\sigma A_\kappa)\frac{\partial(\partial^\sigma A^\kappa)}{\partial(\partial^\nu A^\mu)}$$
$$= -j_\mu + \partial^\nu(\partial_\sigma A_\kappa)\delta_\nu^\sigma \delta_\mu^\kappa = -j_\mu + \partial^\nu \partial_\nu A_\mu \overset{!}{=} 0$$
$$\Leftrightarrow \quad \partial_\nu \partial^\nu A^\mu = j^\mu.$$

6.1.6 Canonical Momentum of Simplified Lagrangian

The canonical momentum now reads

$$\Pi_\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\nu} = -\frac{1}{2}\frac{\partial}{\partial \dot{A}^\nu} = -(\partial_\mu A_\sigma)\frac{\partial}{\partial(\partial^0 A^\nu)}(\partial^\mu A^\sigma) = -(\partial_0 A_\sigma)\frac{\partial}{\partial(\partial^0 A^\nu)}(\partial^0 A^\sigma) = -(\partial_0 A_\sigma)\delta_\nu^\sigma$$
$$= -\partial_0 A_\nu = -\dot{A}_\nu.$$

6.2 Quantization of the EM Field

6.2.1 Expression for Ladder Operators

If we plug in the expansion

$$A^\mu(x) = \int d\tilde{p}\,\left(a_{\lambda p}^\dagger \varepsilon_{\lambda p}^\mu e^{ip\cdot x} + a_{\lambda p}\varepsilon_{\lambda p}^\mu e^{-ip\cdot x}\right),$$

we can proof that the following expressions for the ladder operators are valid *for physical polarizations* ($g\overleftrightarrow{\partial}_\mu f \coloneqq g\partial_\mu f - f\partial_\mu g$):

$$a_{\lambda p}^\dagger = i\varepsilon_{\lambda p}\cdot \int d^3x\, e^{-ip\cdot x}\overleftrightarrow{\partial}_0 A(x) = i\varepsilon_{\lambda p}\cdot\int d^3x\, e^{-ip\cdot x}\left(\dot{A}(x) + i\omega_p A(x)\right)$$
$$= i\varepsilon_{\lambda p}\cdot\int d\tilde{p}'\, d^3x\,\left(i\omega_{p'}\left(a_{\lambda' p'}^\dagger \varepsilon_{\lambda' p'} e^{i(p'-p)\cdot x} - a_{\lambda' p'}\varepsilon_{\lambda' p'} e^{-i(p'+p)\cdot x}\right)\right.$$
$$\left. + i\omega_p\left(a_{\lambda' p'}^\dagger \varepsilon_{\lambda' p'} e^{i(p'-p)\cdot x} + a_{\lambda' p'}\varepsilon_{\lambda' p'} e^{-i(p'+p)\cdot x}\right)\right)$$
$$= i\varepsilon_{\lambda p}\cdot\int d\tilde{p}'\,(2\pi)^3\left(i\omega_{p'}\left(a_{\lambda' p'}^\dagger \varepsilon_{\lambda' p'}\delta(\vec{p}-\vec{p}') - a_{\lambda' p'}\varepsilon_{\lambda' p'} e^{-i2\omega_p t}\delta(\vec{p}+\vec{p}')\right)\right.$$
$$\left. + i\omega_p\left(a_{\lambda' p'}^\dagger \varepsilon_{\lambda' p'}\delta(\vec{p}-\vec{p}') + a_{\lambda' p'}\varepsilon_{\lambda' p'} e^{-i2\omega_p t}\delta(\vec{p}+\vec{p}')\right)\right)$$
$$= -\frac{1}{2}\varepsilon_{\lambda p}\cdot\left(a_{\lambda' p}^\dagger \varepsilon_{\lambda' p} - a_{\lambda',-p}\varepsilon_{\lambda',-p}e^{-i2\omega_p t} + a_{\lambda' p}^\dagger \varepsilon_{\lambda' p} + a_{\lambda',-p}\varepsilon_{\lambda',-p}e^{-i2\omega_p t}\right)$$
$$= -a_{\lambda' p}^\dagger \varepsilon_{\lambda p}\cdot\varepsilon_{\lambda' p} = -a_{\lambda' p}^\dagger \eta_{\lambda\lambda'}.$$

Obviously, for physical polarizations $\lambda = 1, 2$ we get a true statement. We find the annihilation operator by Hermitian conjugation:

$$a_{\lambda p} = -i\varepsilon_{\lambda p}\cdot\int d^3x\, e^{ip\cdot x}\overleftrightarrow{\partial}_0 A(x).$$

6.3 Choosing the Polarization Vectors

6.3.1 Sum over the Physical Polarizations only

From the completeness relation

$$\eta^{\mu\nu} = \eta^{\lambda\lambda'}\varepsilon^{\mu}_{\lambda p}\varepsilon^{\nu}_{\lambda' p} = \varepsilon^{\mu}_{0p}\varepsilon^{\nu}_{0p} - \sum_{\lambda=1,2}\varepsilon^{\mu}_{\lambda p}\varepsilon^{\nu}_{\lambda p} - \varepsilon^{\mu}_{3p}\varepsilon^{\nu}_{3p}$$

we find, by plugging in $\varepsilon^{\mu}_{0p} = n^{\mu}$ and $\varepsilon^{\mu}_{3p} = p^{\mu}/(p\cdot n) - n^{\mu}$:

$$\begin{aligned}\sum_{\lambda=1,2}\varepsilon^{\mu}_{\lambda p}\varepsilon^{\nu}_{\lambda p} &= -\eta^{\mu\nu} + \varepsilon^{\mu}_{0p}\varepsilon^{\nu}_{0p} - \varepsilon^{\mu}_{3p}\varepsilon^{\nu}_{3p} = -\eta^{\mu\nu} + n^{\mu}n^{\nu} - \left(\frac{p^{\mu}}{p\cdot n} - n^{\mu}\right)\left(\frac{p^{\nu}}{p\cdot n} - n^{\nu}\right)\\ &= -\eta^{\mu\nu} + n^{\mu}n^{\nu} - \left(\frac{p^{\mu}}{p\cdot n}\frac{p^{\nu}}{p\cdot n} - n^{\mu}\frac{p^{\nu}}{p\cdot n} - \frac{p^{\mu}}{p\cdot n}n^{\nu} + n^{\mu}n^{\nu}\right)\\ &= -\eta^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{(p\cdot n)^2} + \frac{n^{\mu}p^{\nu} + p^{\mu}n^{\nu}}{p\cdot n}.\end{aligned}$$

6.4 Commutator Relations

6.4.1 Commutator Relations of the Ladder Operators

We found that the conjugate momentum can be given as

$$\Pi^{\mu} = -\dot{A}^{\mu} = -i\int d\tilde{p}\ \omega_p\left(a^{\dagger}_{\lambda p}\varepsilon^{\mu}_{\lambda p}e^{ip\cdot x} - a_{\lambda p}\varepsilon^{\mu}_{\lambda p}e^{-ip\cdot x}\right).$$

If we now plug the field expansions into our commutator relation we get the correct result, if we use the commutator relation

$$\left[a_{\lambda p}, a^{\dagger}_{\lambda' p'}\right] = -(2\pi)^3 2\omega_p \eta_{\lambda\lambda'}\delta(\vec{p} - \vec{p}').$$

of the ladder operators (thereby we want to show, that those are alright). We also need the completeness of the polarization vectors $\eta^{\lambda\lambda'}\varepsilon^{\mu}_{\lambda p}\varepsilon^{\nu}_{\lambda' p} = \eta^{\mu\nu}$:

$$\begin{aligned}[A^{\mu}(\vec{x}), \Pi^{\nu}(\vec{y})] &= -\left[A^{\mu}(\vec{x}), \dot{A}^{\nu}(\vec{y})\right]\\ &= -i\int d\tilde{p}\, d\tilde{p}'\ \left[\left(a^{\dagger}_{\lambda p}\varepsilon^{\mu}_{\lambda p}e^{ip\cdot x} + a_{\lambda p}\varepsilon^{\mu}_{\lambda p}e^{-ip\cdot x}\right), \left(a^{\dagger}_{\lambda' p'}\varepsilon^{\nu}_{\lambda' p'}e^{ip'\cdot y} - a_{\lambda' p'}\varepsilon^{\nu}_{\lambda' p'}e^{-ip'\cdot y}\right)\right]\\ &= -i\int d\tilde{p}\, d\tilde{p}'\ \left(-e^{ip\cdot x}e^{-ip'\cdot y}\varepsilon^{\mu}_{\lambda p}\varepsilon^{\nu}_{\lambda' p'}\left[a^{\dagger}_{\lambda p}, a_{\lambda' p'}\right] + e^{-ip\cdot x}e^{ip'\cdot y}\varepsilon^{\mu}_{\lambda p}\varepsilon^{\nu}_{\lambda' p'}\left[a_{\lambda p}, a^{\dagger}_{\lambda' p'}\right]\right)\\ &= i\int d\tilde{p}\ \left(e^{ip\cdot(x-y)}\varepsilon^{\mu}_{\lambda p}\varepsilon^{\nu}_{\lambda p} + e^{-ip\cdot(x-y)}\varepsilon^{\mu}_{\lambda p}\varepsilon^{\nu}_{\lambda p}\right) = i\eta^{\mu\nu}\delta(\vec{x} - \vec{y}).\end{aligned}$$

Note, that this is the *equal-time* commutator, as indicated by writing $A^{\mu}(\vec{x})$ instead of $A^{\mu}(x)$.

6.5 The Four-Momentum Operator

6.5.1 Calculating the Four-Momentum Operator

Consider our simplified Lagrangian for the Sourceless case $j_{\mu} = 0$:

$$\mathcal{L} = -\frac{1}{2}(\partial_{\mu}A_{\nu})(\partial^{\mu}A^{\nu}).$$

reads

$$\begin{aligned}\mathcal{T}^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\sigma})}(\partial^{\nu}A_{\sigma}) - \mathcal{L}\eta^{\mu\nu} = -(\partial^{\kappa}A^{\eta})\frac{\partial(\partial_{\kappa}A_{\eta})}{\partial(\partial_{\mu}A_{\sigma})}(\partial^{\nu}A_{\sigma}) - \mathcal{L}\eta^{\mu\nu}\\ &= -(\partial^{\mu}A^{\sigma})(\partial^{\nu}A_{\sigma}) + \frac{1}{2}(\partial_{\sigma}A_{\kappa})(\partial^{\sigma}A^{\kappa})\eta^{\mu\nu}\end{aligned}$$

and we have the four-momentum

$$P^\nu = \int d^3x\, \mathcal{T}^{0\nu} = \int d^3x\, \left(-\dot{A}^\sigma(\partial^\nu A_\sigma) + \frac{1}{2}(\partial_\sigma A_\kappa)(\partial^\sigma A^\kappa)\eta^{0\nu}\right).$$

The second term is always zero, since the two derivatives bring down a factor $p_\sigma p'^\sigma$ which is turned into a $\pm p_\sigma p^\sigma$ by δ-functions originating from the integrals

$$\int d^3x\, e^{\pm i(p'\pm p)\cdot x}.$$

Since the EM field is massless, $p_\sigma p^\sigma = \omega_p^2 - \vec{p}^2 = m^2 = 0$ vanishes. What remains is the first term:

$$\begin{aligned}
P^\nu &= -\int d^3x\, \dot{A}^\sigma(\partial^\nu A_\sigma) \\
&= -\int d^3x\, d\tilde{p}\, d\tilde{p}'\, i\omega_p\big(a^\dagger_{\lambda p}\varepsilon^\sigma_{\lambda p}e^{ip\cdot x} \\
&- a_{\lambda p}\varepsilon^\sigma_{\lambda p}e^{-ip\cdot x}\big)\left(ip'^\nu\left(a^\dagger_{\lambda' p'}\varepsilon_{\lambda' p' \sigma}e^{ip'\cdot x} - a_{\lambda' p'}\varepsilon_{\lambda' p' \sigma}e^{-ip'\cdot x}\right)\right) \\
&= \int d^3x\, d\tilde{p}\, d\tilde{p}'\, \omega_p p'^\nu\left(a^\dagger_{\lambda p}a^\dagger_{\lambda' p'}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p' \sigma}e^{i(p'+p)\cdot x} - a_{\lambda p}a^\dagger_{\lambda' p'}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p' \sigma}e^{i(p'-p)\cdot x}\right. \\
&\left.- a^\dagger_{\lambda p}a_{\lambda' p'}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p' \sigma}e^{-i(p'-p)\cdot x} + a_{\lambda p}a_{\lambda' p'}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p' \sigma}e^{-i(p'+p)\cdot x}\right) \\
&= \int d\tilde{p}\, d\tilde{p}'\, \omega_p p'^\nu(2\pi)^3\left(a^\dagger_{\lambda p}a^\dagger_{\lambda' p'}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p' \sigma}\delta(\vec{p}'+\vec{p})e^{i2\omega_p t} - a_{\lambda p}a^\dagger_{\lambda' p'}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p' \sigma}\delta(\vec{p}'-\vec{p})\right. \\
&\left.- a^\dagger_{\lambda p}a_{\lambda' p'}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p' \sigma}\delta(\vec{p}'-\vec{p}) + a_{\lambda p}a_{\lambda' p'}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p' \sigma}\delta(\vec{p}'+\vec{p})e^{-i2\omega_p t}\right) \\
&= \int d\tilde{p}\, \frac{\omega_p p^\nu}{2\omega_p}\left(-a^\dagger_{\lambda p}a^\dagger_{\lambda',-p}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda',-p,\sigma}e^{i2\omega_p t} - a_{\lambda p}a^\dagger_{\lambda' p}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p \sigma} - a^\dagger_{\lambda p}a_{\lambda' p}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda' p \sigma}\right. \\
&\left.- a_{\lambda p}a_{\lambda',-p}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda',-p,\sigma}e^{-i2\omega_p t}\right) \\
&= -\int d\tilde{p}\, \frac{\omega_p p^\nu}{2\omega_p}\left(a_{\lambda p}a^\dagger_{\lambda' p}\eta_{\lambda\lambda'} + a^\dagger_{\lambda p}a_{\lambda' p}\eta_{\lambda\lambda'} + a^\dagger_{\lambda p}a^\dagger_{\lambda',-p}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda',-p,\sigma}e^{i2\omega_p t}\right. \\
&\left.+ a_{\lambda p}a_{\lambda',-p}\varepsilon^\sigma_{\lambda p}\varepsilon_{\lambda',-p,\sigma}e^{-i2\omega_p t}\right).
\end{aligned}$$

Now the last two terms vanish due to antisymmetry when taking $p \to -p$. Note that we can rename the summation indices $\lambda \leftrightarrow \lambda'$. We also throw away the infinite constant (i.e. the commutator) as usual and are left with

$$P^\nu = -\int d\tilde{p}\, \frac{\omega_p p^\nu}{2\omega_p}\eta_{\lambda\lambda'}\left(2a^\dagger_{\lambda p}a_{\lambda' p} + \left[a_{\lambda p}, a^\dagger_{\lambda' p}\right]\right) = -\int d\tilde{p}\, p^\nu\, \eta_{\lambda\lambda'}a^\dagger_{\lambda p}a_{\lambda' p}.$$

6.6 Operators Acting on States

6.6.1 Commutator of the Four-Momentum Operator and the Creation Operator

$$\begin{aligned}
\left[P^\mu, a^\dagger_{\lambda p}\right] &= -\int d\tilde{p}'\, p'^\nu\, \eta^{\lambda'\lambda''}\left[a^\dagger_{\lambda' p'}a_{\lambda'' p'}, a^\dagger_{\lambda p}\right] \\
&= -\int d\tilde{p}'\, p'^\nu\, \eta^{\lambda'\lambda''}\left(a^\dagger_{\lambda' p'}\left[a_{\lambda'' p'}, a^\dagger_{\lambda p}\right] + \left[a^\dagger_{\lambda' p'}, a^\dagger_{\lambda p}\right]a_{\lambda'' p'}\right) \\
&= -\int d\tilde{p}'\, p'^\nu\, \eta^{\lambda'\lambda''}a^\dagger_{\lambda' p'}\left(-(2\pi)^3 2\omega_p\eta_{\lambda''\lambda}\delta\left(\vec{p}-\vec{p}'\right)\right) = p^\nu\, \eta^{\lambda'}_{\lambda}a^\dagger_{\lambda' p} = p^\nu a^\dagger_{\lambda p}.
\end{aligned}$$

The commutator with $a_{\lambda p}$ is got by applying $\dagger$ to this one.

6.7 Gupta-Bleuler Method

6.7.1 Why $\boldsymbol{\partial_\mu A^\mu|\psi\rangle = 0}$ is no good Condition

If we adopt the decomposition $A^\mu = A^{+\mu} + A^{-\mu}$, where

$$A^{+\mu}(x) = \int d\tilde{p}\, a_{\lambda p} \varepsilon^{\mu}_{\lambda p} e^{-ip\cdot x}, \quad A^{-\mu}(x) = \int d\tilde{p}\, a^{\dagger}_{\lambda p} \varepsilon^{\mu}_{\lambda p} e^{ip\cdot x},$$

we find that not even the vacuum state would obey our condition for physical states:

$$\partial_\mu A^\mu |0\rangle = \partial_\mu \underbrace{A^{+\mu}|0\rangle}_{=0} + \underbrace{\partial_\mu A^{-\mu}|0\rangle}_{\neq 0} \neq 0,$$

because $A^{-\mu}$ contains the creation operator (also after derivation).

6.7.2 Calculation of $\boldsymbol{\partial_\mu A^{+\mu}}$

We constructed and chose our polarization vectors such that

$$p \cdot \varepsilon_{\lambda p} = \begin{cases} p \cdot n, & \lambda = 0 \\ 0, & \lambda = 1, 2, \\ -p \cdot n, & \lambda = 3 \end{cases}$$

using $p^2 = 0$. Thus, we can write

$$\partial_\mu A^{+\mu} = \partial_\mu \int d\tilde{p}\, a_{\lambda p} \varepsilon^{\mu}_{\lambda p} e^{-ip\cdot x} = -i \int d\tilde{p}\, p_\mu\, a_{\lambda p} \varepsilon^{\mu}_{\lambda p} e^{-ip\cdot x} = -i \int d\tilde{p}\, e^{-ip\cdot x} (p \cdot n)\big(a_{0p} - a_{3p}\big).$$

6.7.3 Physical State Expectation Value of the Four-Momentum

Using

$$a_{0p}|\psi\rangle = a_{3p}|\psi\rangle, \quad \langle\psi|a^{\dagger}_{0p} = \langle\psi|a^{\dagger}_{3p}$$

we get

$$\begin{aligned}
\langle\psi|P^\nu|\psi\rangle &= -\int d\tilde{p}\, p^\nu\, \eta^{\lambda\lambda'} \langle\psi|a^{\dagger}_{\lambda p} a_{\lambda' p}|\psi\rangle \\
&= \int d\tilde{p}\, p^\nu\, \langle\psi|\big(-a^{\dagger}_{0p}a_{0p} + a^{\dagger}_{1p}a_{1p} + a^{\dagger}_{2p}a_{2p} + a^{\dagger}_{3p}a_{3p}\big)|\psi\rangle \\
&= \int d\tilde{p}\, p^\nu\, \langle\psi|\big(-a^{\dagger}_{0p}a_{0p} + a^{\dagger}_{1p}a_{1p} + a^{\dagger}_{2p}a_{2p} + a^{\dagger}_{0p}a_{0p}\big)|\psi\rangle \\
&= -\sum_{\lambda=1,2} \int d\tilde{p}\, p^\nu\, \eta^{\lambda\lambda'} \langle\psi|a^{\dagger}_{\lambda p} a_{\lambda p}|\psi\rangle.
\end{aligned}$$

6.7.4 Physical States have only positive Norms

We saw that the state $|0, p\rangle = a^{\dagger}_{0p}|0\rangle$ was the one which had a negative norm $\langle 0, p|0, p\rangle < 0$. We will now show that this state does not fulfill the physical state condition, which was shown to be equivalent to $\big(a_{0p} - a_{3p}\big)|\psi\rangle = 0$:

$$\begin{aligned}
\big(a_{0p'} - a_{3p'}\big)|0, p\rangle &= \big(a_{0p'} - a_{3p'}\big)a^{\dagger}_{0p}|0\rangle = \big(a^{\dagger}_{0p}a_{0p'} + \big[a_{0p'}, a^{\dagger}_{0p}\big] - a^{\dagger}_{0p}a_{3p'} - \big[a_{3p'}, a^{\dagger}_{0p}\big]\big)|0\rangle \\
&= -(2\pi)^3 2\omega_p \delta(\vec{p} - \vec{p}')(\eta_{00} - \eta_{30})|0\rangle = -(2\pi)^3 2\omega_p \delta(\vec{p} - \vec{p}')|0\rangle \neq 0.
\end{aligned}$$

6.8 Causality and Propagators

6.8.1 Deriving $\boldsymbol{\widehat{\Delta}(z)}$

Using the completeness relation $\eta^{\lambda\lambda'} \varepsilon^{\mu}_{\lambda p} \varepsilon^{\nu}_{\lambda' p} = \eta^{\mu\nu}$:

$$\begin{aligned}
\widehat{\Delta}(x-y) &:= [A^\mu(x), A^\nu(y)] \\
&= \int d\tilde{p}\, d\tilde{p}'\ \left(\left[a^\dagger_{\lambda p}\varepsilon^\mu_{\lambda p}e^{ip\cdot x} + a_{\lambda p}\varepsilon^\mu_{\lambda p}e^{-ip\cdot x}, a^\dagger_{\lambda' p'}\varepsilon^\nu_{\lambda' p'}e^{ip'\cdot y} + a_{\lambda' p'}\varepsilon^\nu_{\lambda' p'}e^{-ip'\cdot y}\right]\right) \\
&= \int d\tilde{p}\, d\tilde{p}'\ \left(e^{ip\cdot x}e^{-ip'\cdot y}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda' p'}\left[a^\dagger_{\lambda p}, a_{\lambda' p'}\right] + e^{-ip\cdot x}e^{ip'\cdot y}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda' p'}\left[a_{\lambda p}, a^\dagger_{\lambda' p'}\right]\right) \\
&= \int d\tilde{p}\, d\tilde{p}'\ (2\pi)^3 2\omega_p \eta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}')\left(e^{ip\cdot x}e^{-ip'\cdot y}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda' p'} - e^{-ip\cdot x}e^{ip'\cdot y}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda' p'}\right) \\
&= \int d\tilde{p}\ \eta^{\mu\nu}\left(e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)}\right) = -\eta^{\mu\nu}\Delta(x-y).
\end{aligned}$$

6.8.2 The Feynman-Propagator

Using the completeness relation $\eta^{\lambda\lambda'}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda' p} = \eta^{\mu\nu}$ as well as (>4.8.3), in the Feynman propagator we have the matrix elements,

$$\begin{aligned}
&\langle 0|A^\mu(x)A^\nu(y)|0\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\ \left\langle 0\middle|\left(a^\dagger_{\lambda p}\varepsilon^\mu_{\lambda p}e^{ip\cdot x} + a_{\lambda p}\varepsilon^\mu_{\lambda p}e^{-ip\cdot x}\right)\left(a^\dagger_{\lambda' p'}\varepsilon^\nu_{\lambda' p'}e^{ip'\cdot y} + a_{\lambda' p'}\varepsilon^\nu_{\lambda' p'}e^{-ip'\cdot y}\right)\middle|0\right\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\ e^{-ip\cdot x}e^{ip'\cdot y}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda' p'}\left\langle 0\middle|a_{\lambda p}a^\dagger_{\lambda' p'}\middle|0\right\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\ e^{-ip\cdot x}e^{ip'\cdot y}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda' p'}\left\langle 0\middle|a^\dagger_{\lambda' p'}a_{\lambda p} + \left[a_{\lambda p}, a^\dagger_{\lambda' p'}\right]\middle|0\right\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\ e^{-ip\cdot x}e^{ip'\cdot y}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda' p'}\left(-(2\pi)^3 2\omega_p\eta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}')\right) = -\eta^{\mu\nu}\int d\tilde{p}\ e^{-ip\cdot(x-y)} \\
&= -\eta^{\mu\nu}\left\langle 0\middle|\phi(x)\phi^\dagger(y)\middle|0\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\langle 0|A^\nu(y)A^\mu(x)|0\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\ \left\langle 0\middle|\left(a^\dagger_{\lambda p}\varepsilon^\nu_{\lambda p}e^{ip\cdot y} + a_{\lambda p}\varepsilon^\nu_{\lambda p}e^{-ip\cdot y}\right)\left(a^\dagger_{\lambda' p'}\varepsilon^\mu_{\lambda' p'}e^{ip'\cdot x} + a_{\lambda' p'}\varepsilon^\mu_{\lambda' p'}e^{-ip'\cdot x}\right)\middle|0\right\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\ e^{-ip\cdot y}e^{ip'\cdot x}\varepsilon^\nu_{\lambda p}\varepsilon^\mu_{\lambda' p'}\left\langle 0\middle|a_{\lambda p}a^\dagger_{\lambda' p'}\middle|0\right\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\ e^{-ip\cdot y}e^{ip'\cdot x}\varepsilon^\nu_{\lambda p}\varepsilon^\mu_{\lambda' p'}\left\langle 0\middle|a^\dagger_{\lambda' p'}a_{\lambda p} + \left[a_{\lambda p}, a^\dagger_{\lambda' p'}\right]\middle|0\right\rangle \\
&= \int d\tilde{p}\, d\tilde{p}'\ e^{-ip\cdot y}e^{ip'\cdot x}\varepsilon^\nu_{\lambda p}\varepsilon^\mu_{\lambda' p'}\left(-(2\pi)^3 2\omega_p\eta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}')\right) = -\eta^{\mu\nu}\int d\tilde{p}\ e^{ip\cdot(x-y)} \\
&= -\eta^{\mu\nu}\left\langle 0\middle|\phi^\dagger(y)\phi(x)\middle|0\right\rangle.
\end{aligned}$$

Thus, we find that we can write

$$\begin{aligned}
\widehat{D}_F(x-y) &= \begin{cases}\langle 0|A^\mu(x)A^\nu(y)|0\rangle, & x^0 \geq y^0 \\ \langle 0|A^\nu(y)A^\mu(x)|0\rangle, & y^0 \geq x^0\end{cases} = -\eta^{\mu\nu}\begin{cases}\left\langle 0\middle|\phi(x)\phi^\dagger(y)\middle|0\right\rangle, & x^0 \geq y^0 \\ \left\langle 0\middle|\phi^\dagger(y)\phi(x)\middle|0\right\rangle, & y^0 \geq x^0\end{cases} \\
&= -\eta^{\mu\nu}D_F(x-y).
\end{aligned}$$

7.2 Interacting Fock Space

7.2.1 Completeness Relation

The completeness relation of the Fock space with the vacuum $|\Omega\rangle$ and the N particle state $|\lambda, \vec{p}\rangle$ with total momentum $\vec{p}$ reads

$$\mathbb{I} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int d\tilde{p}_\lambda \, |\lambda, \vec{p}\rangle\langle\lambda, \vec{p}|.$$

The sum over λ is formal and includes integral over continuous parameters like the relative momenta. The energy ω_p inside the measure $d\tilde{p}$ is in this case defined as $\omega_{p\lambda}^2 = \vec{p}^2 + m_\lambda^2$, where m_λ is the invariant mass of all the particles,

$$m_\lambda^2 = \left(\sum_{i=1}^N p_i\right)^2,$$

p_i being the four-momentum of particle i. Thus, the measure $d\tilde{p}$ depends on the configuration of the N particles and thereby on λ, which is why $d\tilde{p}_\lambda$ carries this index.

7.2.2 Vacuum–Momentum State Matrix Element

For a general Lorentz transformation $U(\Lambda, a)$, we assume axiomatically

$$U(\Lambda, a)\phi(x)U^{-1}(\Lambda, a) = \phi(\Lambda x + a).$$

Furthermore, we now from section 2.4 that simple translation can be written as $U(1, a) = \exp(-ia \cdot P)$ with momentum operator P. The vacuum state $|\Omega\rangle$ is invariant under Lorentz transformation. Let $U(\Lambda_{\vec{p}})$ be a Lorentz boost which takes the rest frame to momentum $\vec{p}$. Then, we find

$$\begin{aligned}\langle\Omega|\phi(x)|\lambda, \vec{p}\rangle &= \langle\Omega|e^{ix\cdot P}\phi(0)e^{-ix\cdot P}|\lambda, \vec{p}\rangle = \langle\Omega|\phi(0)|\lambda, \vec{p}\rangle e^{-ix\cdot p} \\ &= \langle\Omega|U(\Lambda_{\vec{p}})\phi(0)U^{-1}(\Lambda_{\vec{p}})|\lambda, 0\rangle e^{-ix\cdot p} = \langle\Omega|\phi(\Lambda_{\vec{p}}0)|\lambda, 0\rangle e^{-ix\cdot p} = \langle\Omega|\phi(0)|\lambda, 0\rangle e^{-ix\cdot p}.\end{aligned}$$

7.3 Källén-Lehmann Spectral Representation

7.3.1 Vacuum Expectation Value of two Fields

Using the fact that we can redefine our fields such that the constant $\langle\Omega|\phi(x)|\Omega\rangle = \langle\Omega|e^{ix\cdot P}\phi(0)e^{-ix\cdot P}|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle$ vanishes, we find

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \underbrace{\langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle}_{=0} + \sum_\lambda \int d\tilde{p}_\lambda \, \langle\Omega|\phi(x)|\lambda, \vec{p}\rangle\langle\lambda, \vec{p}|\phi(y)|\Omega\rangle \\ &= \sum_\lambda \int d\tilde{p}_\lambda \, |\langle\Omega|\phi(0)|\lambda, \vec{p}\rangle|^2 e^{-ip\cdot(x-y)} = \sum_\lambda Z_\lambda \int d\tilde{p}_\lambda \, e^{-ip\cdot(x-y)}.\end{aligned}$$

Except for the sum over λ and the Z_λ, this is the same formula as we had in section 4.8 for the free vacuum expectation value $\langle 0|\phi(x)\phi(y)|0\rangle$. The corresponding *time-ordered* expectation value was the Feynman propagator D_F. Thus, the interacting time-ordered vacuum matrix element can be given as a sum of free Feynman propagators:

$$\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle = \mathcal{T}\sum_\lambda Z_\lambda \int d\tilde{p}_\lambda\, e^{-ip\cdot(x-y)} = \mathcal{T}\sum_\lambda Z_\lambda\, \langle 0|\phi(x)\phi(y)|0\rangle_\lambda$$
$$= \sum_\lambda Z_\lambda\, \langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle_\lambda = \sum_\lambda Z_\lambda\, D_F\big(x-y, m_\lambda^2\big).$$

7.3.2 The Mass Spectrum

We can write

$$\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty dM^2\, \rho(M^2)\, D_F(x-y, M^2),$$

where

$$\rho(M^2) = \sum_\lambda Z_\lambda\, \delta\big(M^2 - m_\lambda^2\big)$$

is the mass spectrum. If we plug this formula for $\rho(M^2)$ in, we easily get our old formula for the time-ordered matrix element back.

7.4 The S-Matrix

7.4.1 Impact of the S-Matrix on "in" and "out" States

Consider some arbitrary state $|\alpha_\pm\rangle$, which is created out of the vacuum by some combination $F_\alpha(\varphi_\pm)$ of fields $\varphi_\pm$. Using $\varphi_\pm = S^{\mp 1}\varphi_\mp S^{\pm 1}$ and $S^{\pm 1}|\Omega\rangle = |\Omega\rangle$, we find

$$|\alpha_\pm\rangle = F_\alpha(\varphi_\pm)|\Omega\rangle = F_\alpha\big(S^{\mp 1}\varphi_\mp S^{\pm 1}\big)|\Omega\rangle = S^{\mp 1}F_\alpha(\varphi_\mp)S^{\pm 1}|\Omega\rangle = S^{\mp 1}F_\alpha(\varphi_\mp)|\Omega\rangle = S^{\mp 1}|\alpha_\mp\rangle.$$

If we take this equation and multiply it with $\langle\alpha_\pm|$, we find

$$1 = \langle\alpha_\pm|S^{\mp 1}|\alpha_\mp\rangle.$$

If we perform Hermitian conjugation on both sides, we find

$$1 = \left\langle\alpha_\mp\middle|\big(S^{\mp 1}\big)^\dagger\middle|\alpha_\pm\right\rangle = \langle\alpha_\pm|(S^{\pm 1})^\dagger|\alpha_\mp\rangle,$$

where we just turned the $\pm$ and $\mp$ sign upside down in the last step, i. e. $\pm \to \mp$ and $\mp \to \pm$. Since those two equations hold for arbitrary "in" and "out" states, we can deduct

$$\langle\alpha_\pm|S^{\mp 1}|\alpha_\mp\rangle = \langle\alpha_\pm|(S^{\pm 1})^\dagger|\alpha_\mp\rangle \qquad \Longrightarrow \qquad S^{-1} = S^\dagger \quad \Leftrightarrow \quad S^\dagger S = 1.$$

7.5 LSZ Reduction

7.5.1 Form of Creation Operator

In 4.4 we had the expression

$$a_{-p}^\dagger = e^{-i\omega_p t}\Big(\omega_p\phi(\vec{p}) - i\Pi(\vec{p})\Big),$$

which holds for *free* scalar fields, and we know that for a scalar field we have $\Pi(x) = \dot{\phi}(x)$. Plugging in $\phi(\vec{p}), \Pi(\vec{p})$ in terms of $\phi(x), \Pi(x)$, i.e. the Fourier transform, we find

$$
\begin{aligned}
a_p^\dagger &= e^{-i\omega_p t}\left(\omega_p \phi(-\vec{p}) - i\Pi(-\vec{p})\right) = e^{-i\omega_p t}\left(\omega_p \phi^\dagger(\vec{p}) - i\Pi^\dagger(\vec{p})\right) \\
&= \int d^3x\, e^{-i\omega_p t} e^{i\vec{x}\vec{p}}\left(\omega_p \phi(\vec{x}) - i\Pi(\vec{x})\right) = \int d^3x\, e^{-i\omega_p t} e^{i\vec{x}\vec{p}}\left(\omega_p \phi(\vec{x}) - i\Pi(\vec{x})\right) \\
&= \int d^3x\, e^{-ix\cdot p}\left(\omega_p \phi(\vec{x}) - i\dot{\phi}(\vec{x})\right) = -i\int d^3x\, e^{-ix\cdot p} \overleftrightarrow{\partial}_0 \phi(\vec{x}),
\end{aligned}
$$

where $g\overleftrightarrow{\partial}_\mu f \coloneqq g\partial_\mu f - f\partial_\mu g$. Note that we wrote $\phi(\vec{x})$ instead of $\phi(x)$ since the integral is only over d^3x, but the time-coordinate is still there implicitly and we can also write $\phi(\vec{x}) = \phi(x)$. Also, we omitted the indices $\pm$ for the "in" and "out" fields/ladder operators. The calculation above just holds for any free field and "in" and "out" fields both meet this condition.

7.5.2 Identities for commuting Ladder Operators

We are now going to show the following four identities:

$$
\begin{aligned}
&a_{+,q} - a_{-,q} = I_q, && a_{+,q}^\dagger - a_{-,q}^\dagger = -I_{-q}, \\
&a_{+,q} I_{p_1,\dots,p_n} - I_{p_1,\dots,p_n} a_{-,q} = I_{p_1,\dots,p_n,q}, && a_{+,q}^\dagger I_{p_1,\dots,p_n} - I_{p_1,\dots,p_n} a_{-,q}^\dagger = -I_{p_1,\dots,p_n,-q}.
\end{aligned}
$$

where

$$
I_{p_1,\dots,p_n} = \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n}\, \mathcal{T}\phi(x_1)\cdots\phi(x_n) \qquad \text{and} \qquad \mathcal{D}_{x,p} = iZ^{-1/2}\, d^4x\, e^{ix\cdot p}(\Box_x + m^2).
$$

Thus, we can commute the "out" ladder operators with the integrals $I_{p_1,\dots,p_n}$ and at the same time turn them into "in" ladder operators. However, while doing so we get an extra term, which is just another integral $I_{p_1,\dots,p_n,q}$ with an additional momentum q. Note that this additional index of $I_{p_1,\dots,p_n,q}$ just gives an additional factor in the integral. If the additional momentum q traces back to the commutation of creation operators, we get this additional momentum with a minus sign, $-q$, and we get a overall minus sign in front of the $I_{p_1,\dots,p_n,-q}$.

Alright, so let's start to proof the identities above and thereby derive the formula for $I_{p_1,\dots,p_n}$:

$$
\begin{aligned}
I_q &= a_{+,q} - a_{-,q} = i\int d^3z\, e^{iz\cdot q}\overleftrightarrow{\partial}_0 \phi_+(z) - i\int d^3z\, e^{iz\cdot q}\overleftrightarrow{\partial}_0\phi_-(z) \\
&= iZ^{-1/2}\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right)\int d^3z\, e^{iz\cdot q}\overleftrightarrow{\partial}_0\phi(z) = iZ^{-1/2}\int_{-\infty}^{\infty} dt\, \partial_0 \int d^3z\, e^{iz\cdot q}\overleftrightarrow{\partial}_0\phi(z) \\
&= iZ^{-1/2}\int d^4z\, \partial_0\, e^{iz\cdot q}\overleftrightarrow{\partial}_0\phi(z) = iZ^{-1/2}\int d^4z\, \partial_0\left(e^{iz\cdot q}\partial_0\phi(z) - \phi(z)\partial_0 e^{iz\cdot q}\right) \\
&= iZ^{-1/2}\int d^4z\left(e^{iz\cdot q}\partial_0^2\phi(z) - \phi(z)\partial_0^2 e^{iz\cdot q}\right) = iZ^{-1/2}\int d^4z\, e^{iz\cdot q}\left(\partial_0^2 + \omega_q^2\right)\phi(z) \\
&= iZ^{-1/2}\int d^4z\, e^{iz\cdot q}(\partial_0^2 + \vec{q}^2 + m^2)\phi(z) = iZ^{-1/2}\int d^4z\, e^{iz\cdot q}\left(\partial_0^2 - \overleftarrow{\nabla}^2 + m^2\right)\phi(z) \\
&= iZ^{-1/2}\int d^4z\, e^{iz\cdot q}(\partial_0^2 - \nabla^2 + m^2)\phi(z) = iZ^{-1/2}\int d^4z\, e^{iz\cdot q}(\Box + m^2)\phi(z) \\
&= \int \mathcal{D}_{z,q}\, \phi(z).
\end{aligned}
$$

Here, $\overleftarrow{\nabla}$ is the nabla operator acting to the left, such that $e^{-ix\cdot p}\left(-\overleftarrow{\nabla}^2\right) = \vec{p}^2$. We then got from this $\overleftarrow{\nabla}$ to the usual right-acting ∇ by integration by parts (twice). The equation for the creation operators is got simply by Hermitian conjugation, using the fact that $I_q^\dagger = -I_{-q}$. Next, we can look at the third identity:

$$
\begin{aligned}
I_{p_1,\dots,p_n,q} &= a_{+,q} I_{p_1,\dots,p_n} - I_{p_1,\dots,p_n} a_{-,q} \\
&= i \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^3 z \left(e^{iz\cdot q} \overleftrightarrow{\partial}_{z_0} \phi_+(z) \mathcal{T}\phi(x_1) \cdots \phi(x_n) - \mathcal{T}\phi(x_1) \cdots \phi(x_n) e^{iz\cdot q} \overleftrightarrow{\partial}_{z_0} \phi_-(z) \right) \\
&= iZ^{-1/2} \left(\lim_{z_0 \to \infty} - \lim_{z_0 \to -\infty} \right) \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^3 z \, e^{iz\cdot q} \overleftrightarrow{\partial}_{z_0} \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z) \\
&= iZ^{-1/2} \int_{-\infty}^{\infty} dz_0 \, \partial_{z_0} \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^3 z \, e^{iz\cdot q} \overleftrightarrow{\partial}_{z_0} \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z) \\
&= iZ^{-1/2} \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^4 z \, \partial_{z_0} e^{iz\cdot q} \overleftrightarrow{\partial}_{z_0} \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z) \\
&= iZ^{-1/2} \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^4 z \, e^{iz\cdot q} (\Box_z + m^2) \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z) \\
&= \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} \mathcal{D}_{z,q} \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z).
\end{aligned}
$$

Here, we used $\partial_{z_0} e^{iz\cdot q} \overleftrightarrow{\partial}_{z_0} f(z) = e^{iz\cdot q} (\Box_z + m^2) f(z)$, which was derived along the way of deriving the formula of I_q above. Finally, what is left is

$$
\begin{aligned}
-I_{p_1,\dots,p_n,-q} &= a^\dagger_{+,q} I_{p_1,\dots,p_n} - I_{p_1,\dots,p_n} a^\dagger_{-,q} \\
&= -i \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^3 z \Big(e^{-iz\cdot q} \overleftrightarrow{\partial}_{z_0} \phi_+(z) \mathcal{T}\phi(x_1) \cdots \phi(x_n) \\
&\qquad\qquad - \mathcal{T}\phi(x_1) \cdots \phi(x_n) e^{-iz\cdot q} \overleftrightarrow{\partial}_{z_0} \phi_-(z) \Big) \\
&= -iZ^{-1/2} \left(\lim_{z_0 \to \infty} - \lim_{z_0 \to -\infty} \right) \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^3 z \, e^{-iz\cdot q} \overleftrightarrow{\partial}_{z_0} \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z) \\
&= -iZ^{-1/2} \int_{-\infty}^{\infty} dz_0 \, \partial_{z_0} \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^3 z \, e^{-iz\cdot q} \overleftrightarrow{\partial}_{z_0} \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z) \\
&= -iZ^{-1/2} \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^4 z \, \partial_{z_0} e^{-iz\cdot q} \overleftrightarrow{\partial}_{z_0} \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z) \\
&= -iZ^{-1/2} \int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} d^4 z \, e^{-iz\cdot q} (\Box_z + m^2) \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z) \\
&= -\int \mathcal{D}_{x_1,p_1} \cdots \mathcal{D}_{x_n,p_n} \mathcal{D}_{z,-q} \, \mathcal{T}\phi(x_1) \cdots \phi(x_n) \phi(z).
\end{aligned}
$$

7.5.3 Disconnected Parts

Using the identities from (>7.5.3), we can show that, for the example of a $2 \to 2$ process,

$$
\begin{aligned}
S_{\beta\alpha} &= \langle\Omega|a_{+,q_1}a_{+,q_2}a^\dagger_{-,p_1}a^\dagger_{-,p_2}|\Omega\rangle = \langle\Omega|a_{+,q_1}(a_{-,q_2}+I_{q_2})a^\dagger_{-,p_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&= \langle\Omega|a_{+,q_1}a_{-,q_2}a^\dagger_{-,p_1}a^\dagger_{-,p_2}|\Omega\rangle + \langle\Omega|a_{+,q_1}I_{q_2}a^\dagger_{-,p_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&= \langle\Omega|a_{+,q_1}\left(a^\dagger_{-,p_1}a_{-,q_2}+\left[a_{-,q_2},a^\dagger_{-,p_1}\right]\right)a^\dagger_{-,p_2}|\Omega\rangle + \langle\Omega|a_{+,q_1}\left(a^\dagger_{+,p_1}I_{q_2}+I_{q_2,-p_1}\right)a^\dagger_{-,p_2}|\Omega\rangle \\
&= \langle\Omega|a_{+,q_1}a^\dagger_{-,p_1}a_{-,q_2}a^\dagger_{-,p_2}|\Omega\rangle + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&\quad + \langle\Omega|a_{+,q_1}a^\dagger_{+,p_1}I_{q_2}a^\dagger_{-,p_2}|\Omega\rangle + \langle\Omega|a_{+,q_1}I_{q_2,-p_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&= \langle\Omega|a_{+,q_1}a^\dagger_{-,p_1}\left(a^\dagger_{-,p_2}a_{-,q_2}+\left[a_{-,q_2},a^\dagger_{-,p_2}\right]\right)|\Omega\rangle + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&\quad + \langle\Omega|\left(a^\dagger_{+,p_1}a_{+,q_1}+\left[a_{+,q_1},a^\dagger_{+,p_1}\right]\right)I_{q_2}a^\dagger_{-,p_2}|\Omega\rangle + \langle\Omega|\left(I_{q_2,-p_1}a_{-,q_1}+I_{q_2,-p_1,q_1}\right)a^\dagger_{-,p_2}|\Omega\rangle \\
&= (2\pi)^3 2\omega_{p_2}\delta(\vec{p}_2-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_1}|\Omega\rangle + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&\quad + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_1)\langle\Omega|I_{q_2}a^\dagger_{-,p_2}|\Omega\rangle + \langle\Omega|I_{q_2,-p_1}a_{-,q_1}a^\dagger_{-,p_2}|\Omega\rangle + \langle\Omega|I_{q_2,-p_1,q_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&= (2\pi)^3 2\omega_{p_2}\delta(\vec{p}_2-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_1}|\Omega\rangle + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&\quad + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_1)\langle\Omega|a^\dagger_{-,p_2}I_{q_2}+I_{q_2,-p_2}|\Omega\rangle \\
&\quad + \langle\Omega|I_{q_2,-p_1}\left(a^\dagger_{-,p_2}a_{-,q_1}+\left[a_{-,q_1},a^\dagger_{-,p_2}\right]\right)|\Omega\rangle + \langle\Omega|a^\dagger_{-,p_2}I_{q_2,-p_1,q_1}+I_{q_2,-p_1,q_1,-p_2}|\Omega\rangle \\
&= (2\pi)^3 2\omega_{p_2}\delta(\vec{p}_2-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_1}|\Omega\rangle + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&\quad + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_1)\langle\Omega|I_{q_2,-p_2}|\Omega\rangle + (2\pi)^3 2\omega_{p_2}\delta(\vec{p}_2-\vec{q}_1)\langle\Omega|I_{q_2,-p_1}|\Omega\rangle \\
&\quad + \langle\Omega|I_{q_2,-p_1,q_1,-p_2}|\Omega\rangle \\
&= (2\pi)^3 2\omega_{p_2}\delta(\vec{p}_2-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_1}|\Omega\rangle + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_2}|\Omega\rangle \\
&\quad + (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_1)\langle\Omega|a_{+,q_2}a^\dagger_{-,p_2}|\Omega\rangle - (2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_1)(2\pi)^3 2\omega_{p_2}\delta(\vec{p}_2-\vec{q}_2) \\
&\quad + (2\pi)^3 2\omega_{p_2}\delta(\vec{p}_2-\vec{q}_1)\langle\Omega|a_{+,q_2}a^\dagger_{-,p_1}|\Omega\rangle - (2\pi)^3 2\omega_{p_2}\delta(\vec{p}_2-\vec{q}_1)(2\pi)^3 2\omega_{p_1}\delta(\vec{p}_1-\vec{q}_1) \\
&\quad + \langle\Omega|I_{q_2,-p_1,q_1,-p_2}|\Omega\rangle.
\end{aligned}
$$

In the last step, we used

$$
\begin{aligned}
\langle\Omega|a_{+,q}a^\dagger_{-,p}|\Omega\rangle &= \langle\Omega|(a_{-,q}+I_q)a^\dagger_{-,p}|\Omega\rangle = \langle\Omega|a_{-,q}a^\dagger_{-,p}|\Omega\rangle + \langle\Omega|I_q a^\dagger_{-,p}|\Omega\rangle \\
&= \langle\Omega|a^\dagger_{-,p}a_{-,q}+\left[a_{-,q},a^\dagger_{-,p}\right]|\Omega\rangle + \langle\Omega|a^\dagger_{+,p}I_q+I_{q,-p}|\Omega\rangle = (2\pi)^3 2\omega_p\delta(\vec{p}-\vec{q}) + \langle\Omega|I_{q,-p}|\Omega\rangle
\end{aligned}
$$

$$
\Leftrightarrow \quad \langle\Omega|I_{q,-p}|\Omega\rangle = -(2\pi)^3 2\omega_p\delta(\vec{p}-\vec{q}) + \langle\Omega|a_{+,q}a^\dagger_{-,p}|\Omega\rangle.
$$

7.5.4 LSZ Reduction for Fermions

We now want to repeat the calculation for the Dirac field. We found in (>5.1.1) that the ladder operator $a_{\alpha p}$ can be given *for free fields* as

$$
a_{\alpha p} = e^{i\omega_p t}\bar{u}_{\alpha p}\gamma^0\psi^-(\vec{p}), \quad \psi^\pm(\vec{p}) = \int d^3x\, e^{\pm i\vec{p}\vec{x}}\psi(\vec{x}),
$$

and from there we easily find the Hermitian conjugate $a^\dagger_{\alpha p}$:

$$
a^\dagger_{\alpha p} = e^{-i\omega_p t}\psi^{-\dagger}(\vec{p})\gamma^0\bar{u}^\dagger_{\alpha p} = e^{-i\omega_p t}\left(\int d^3x\, e^{i\vec{p}\vec{x}}\psi^\dagger(\vec{x})\right)u_{\alpha p} = \int d^3x\, e^{-ip\cdot x}\,\bar{\psi}(\vec{x})\gamma^0 u_{\alpha p}.
$$

Using this expression for the "in" and "out" ladder operators, we now can calculate[1]

[1] For fermions, we need to introduce a new factor Z and we call it Z_2.

$$
\begin{aligned}
a^\dagger_{+,\alpha q} - a^\dagger_{-,\alpha q} &= \int d^3x\, e^{-ix\cdot q}\bar\psi_+(x)\gamma^0 u_{\alpha q} - \int d^3x\, e^{-ix\cdot q}\bar\psi_-(x)\gamma^0 u_{\alpha q} \\
&= Z_2^{-1/2}\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right)\int d^3x\, e^{-ix\cdot q}\bar\psi(x)\gamma^0 u_{\alpha q} = Z_2^{-1/2}\int_{-\infty}^{\infty} dt\,\partial_0 \int d^3x\, e^{-ix\cdot q}\bar\psi(x)\gamma^0 u_{\alpha q} \\
&= Z_2^{-1/2}\int d^4x\,\partial_0 e^{-ix\cdot q}\bar\psi(x)\gamma^0 u_{\alpha q} = Z_2^{-1/2}\int d^4x\,\left(e^{-ix\cdot q}\partial_0\bar\psi(x) + \bar\psi(x)\partial_0 e^{-ix\cdot q}\right)\gamma^0 u_{\alpha q} \\
&= Z_2^{-1/2}\int d^4x\,\bar\psi(x)\left(\overleftarrow{\partial}_0\gamma^0 + \partial_0\gamma^0\right)u_{\alpha q}e^{-ix\cdot q} = Z_2^{-1/2}\int d^4x\,\bar\psi(x)\left(\overleftarrow{\partial}_0\gamma^0 - iq_0\gamma^0\right)u_{\alpha q}e^{-ix\cdot q} \\
&= Z_2^{-1/2}\int d^4x\,\bar\psi(x)\left(\overleftarrow{\partial}_0\gamma^0 + iq_i\gamma^i - im\right)u_{\alpha q}e^{-ix\cdot q} \\
&= Z_2^{-1/2}\int d^4x\,\bar\psi(x)\left(\overleftarrow{\partial}_0\gamma^0 - \partial_i\gamma^i - im\right)u_{\alpha q}e^{-ix\cdot q} \\
&= Z_2^{-1/2}\int d^4x\,\bar\psi(x)\left(\overleftarrow{\partial}_0\gamma^0 + \overleftarrow{\partial}_i\gamma^i - im\right)u_{\alpha q}e^{-ix\cdot q} = Z_2^{-1/2}\int d^4x\,\bar\psi(x)\left(\overleftarrow{\not\partial} - im\right)u_{\alpha q}e^{-ix\cdot q} \\
&= -iZ_2^{-1/2}\int d^4x\,\bar\psi(x)\left(i\overleftarrow{\not\partial} + m\right)u_{\alpha q}e^{-ix\cdot q}.
\end{aligned}
$$

The sudden appearance of the mass m inside the brackets comes from the fact that[1] $(\not q - m)u_{\alpha q} = (\gamma^0 q_0 + \gamma^i q_i - m)u_{\alpha q} = 0$. Furthermore, we got from $-\partial_i$ to $\overleftarrow{\partial}_i$ by means of integration by parts. By Hermitian conjugation we find, using $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$,

$$
\begin{aligned}
a_{+,\alpha q} - a_{-,\alpha q} &= iZ_2^{-1/2}\int d^4x\,\left(\bar\psi(x)\left(i\overleftarrow{\not\partial} + m\right)u_{\alpha q}\right)^\dagger e^{iq\cdot x} \\
&= iZ_2^{-1/2}\int d^4x\, e^{iq\cdot x}u^\dagger_{\alpha q}(-i\gamma^0\not\partial\gamma^0 + m\gamma^0\gamma^0)\gamma^0\psi(x) \\
&= -iZ_2^{-1/2}\int d^4x\, e^{iq\cdot x}\bar u_{\alpha q}(i\not\partial - m)\psi(x).
\end{aligned}
$$

We now could also define Integral $I_{p_1,\dots,p_n}$ and derive their formulas and commutation behaviors just in the same way as in the scalar field case in (>7.5.2). Unfortunately, to find a neat notation for the $I_{p_1,\dots,p_n}$'s would be a little bit more challenging, because now the differential operators $\left(i\overleftarrow{\not\partial} + m\right)$ and $(i\not\partial - m)$ have to be place on the right- and left-hand side of the fields respectively. Nevertheless, it should be clear what happens and it should be clear that just in the same way as for the scalar field in (>7.5.3) we can commute the creation operators in a vacuum matrix element to the left and the annihilation operators to the right, produce a lot of disconnected parts, throw them away and stay with the single interacting matrix element. For example, for a $2 \to 2$ process we end up with

$$
\begin{aligned}
S_{\beta\alpha} &= \langle\Omega|a_{+,\alpha_1 q_1}a_{+,\alpha_2 q_2}a^\dagger_{-,\alpha_3 q_3}a^\dagger_{-,\alpha_4 q_4}|\Omega\rangle \\
&= \text{d. p.} + (-i)^2 i^2 Z_2^{-2}\int d^4x_1 d^4x_2 d^4x_3 d^4x_3\; e^{iq_1\cdot x_1}e^{iq_2\cdot x_2}\;\bar u_{\alpha_1 q_1}\bar u_{\alpha_2 q_2}\left(i\not\partial_{x_1} - m\right)\left(i\not\partial_{x_2} - m\right) \\
&\qquad \langle\Omega|\mathcal{T}\psi(x_1)\psi(x_2)\bar\psi(x_3)\bar\psi(x_4)|\Omega\rangle\left(i\overleftarrow{\not\partial}_{x_3} + m\right)\left(i\overleftarrow{\not\partial}_{x_4} + m\right)u_{\alpha_3 q_3}u_{\alpha_4 q_4}\; e^{-ix_3\cdot q_3}e^{-ix_4\cdot q_4}.
\end{aligned}
$$

Note, that for the creation operators the replacement equation read $a^\dagger_{+,\alpha q} - a^\dagger_{-,\alpha q} = -iZ^{-1/2}\int\cdots$. In the matrix element $S_{\beta\alpha}$ the creation operators as "in" operators (with a minus index). Thus, their contribution to the prefactor is $+iZ^{-1/2}$.

7.5.5 LSZ Reduction for Anti-Fermions

For antifermions we found

$$
b^\dagger_{\alpha p} = e^{-i\omega_p t}\bar v_{\alpha p}\gamma^0\psi^+(\vec p), \quad \psi^+(\vec p) := \int d^3x\, e^{+i\vec p\vec x}\psi(\vec x).
$$

And now perform again the completely analogous calculation as for Fermions in (>7.5.4):

[1] Do not get confused with the minus signs in the contraction. It is $\gamma^\mu p_\mu = \gamma^0 p_0 + \gamma^i p_i = \gamma^0 p^0 - \vec\gamma\vec p$.

$$
\begin{aligned}
b^\dagger_{+,\alpha q} - b^\dagger_{-,\alpha q} &= \int d^3x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\gamma^0\psi_+(x) - \int d^3x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\gamma^0\psi_-(x) \\
&= Z_2^{-1/2}\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right)\int d^3x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\gamma^0\psi(x) = Z_2^{-1/2}\int_{-\infty}^{\infty} dt\,\partial_0 \int d^3x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\gamma^0\psi(x) \\
&= Z_2^{-1/2}\int d^4x\,\partial_0\, e^{-ix\cdot q}\bar{v}_{\alpha q}\gamma^0\psi(x) = Z_2^{-1/2}\int d^4x\,\bar{v}_{\alpha q}\gamma^0\left(\psi(x)\partial_0 e^{-ix\cdot q} + e^{-ix\cdot q}\partial_0\psi(x)\right) \\
&= Z_2^{-1/2}\int d^4x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\left(\overleftarrow{\partial}_0\gamma^0 + \partial_0\gamma^0\right)\psi(x) \\
&= Z_2^{-1/2}\int d^4x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\left(-iq_0\gamma^0 + \partial_0\gamma^0\right)\psi(x) \\
&= Z_2^{-1/2}\int d^4x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\left(iq_i\gamma^i + im + \partial_0\gamma^0\right)\psi(x) \\
&= Z_2^{-1/2}\int d^4x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\left(-\overleftarrow{\partial}_i\gamma^i + im + \partial_0\gamma^0\right)\psi(x) \\
&= Z_2^{-1/2}\int d^4x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\left(\partial_i\gamma^i + im + \partial_0\gamma^0\right)\psi(x) = Z_2^{-1/2}\int d^4x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\left(\not\partial + im\right)\psi(x) \\
&= -iZ_2^{-1/2}\int d^4x\, e^{-ix\cdot q}\bar{v}_{\alpha q}\left(i\not\partial - m\right)\psi(x).
\end{aligned}
$$

In the step where the mass appears, we used $\bar{v}_{\alpha p}(\not p + m) = \bar{v}_{\alpha p}\left(\gamma^0 p_0 + \gamma^i p_i + m\right) = 0$. Again, the $b^\dagger_{-,\alpha q}$ appears in matrix elements, not the $b^\dagger_{+,\alpha q}$. Thus, the correct prefactor in the LZS reduction formula is $+iZ^{-1/2}$. By Hermitian conjugation we find, using $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$,

$$
\begin{aligned}
b_{+,\alpha q} - b_{-,\alpha q} &= iZ_2^{-1/2}\int d^4x\, e^{iq\cdot x}\left(\bar{v}_{\alpha q}(i\not\partial - m)\psi(x)\right)^\dagger \\
&= iZ_2^{-1/2}\int d^4x\,\psi^\dagger(x)\left(-i\gamma^0\overleftarrow{\not\partial}\gamma^0 - m\gamma^0\gamma^0\right)\gamma^0 v_{\alpha q}e^{iq\cdot x} \\
&= -iZ_2^{-1/2}\int d^4x\,\bar{\psi}(x)\left(i\overleftarrow{\not\partial} + m\right)v_{\alpha q}e^{iq\cdot x}.
\end{aligned}
$$

7.5.6 LSZ Reduction for Photons

For photons we found

$$
a^\dagger_{\lambda p} = i\varepsilon_{\lambda p}\cdot\int d^3x\, e^{-ip\cdot x}\overleftrightarrow{\partial}_0 A(x), \quad a_{\lambda p} = -i\varepsilon_{\lambda p}\cdot\int d^3x\, e^{ip\cdot x}\overleftrightarrow{\partial}_0 A(x), \quad g\overleftrightarrow{\partial}_\mu f = g\partial_\mu f - f\partial_\mu g.
$$

We are going to use that for the photon momentum we have $q_0^2 = \vec{q}$:

$$
\begin{aligned}
a^\dagger_{+,\lambda q} - a^\dagger_{-,\lambda q} &= i\varepsilon_{\lambda q}\cdot\int d^3x\, e^{-ix\cdot q}\overleftrightarrow{\partial}_0 A_+(x) - i\varepsilon_{\lambda q}\cdot\int d^3x\, e^{-ix\cdot q}\overleftrightarrow{\partial}_0 A_-(x) \\
&= Z_3^{-1/2}\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right) i\varepsilon_{\lambda q}\cdot\int d^3x\, e^{-ix\cdot q}\overleftrightarrow{\partial}_0 A(x) \\
&= iZ_3^{-1/2}\varepsilon_{\lambda q}\cdot\int_{-\infty}^{\infty} dt\,\partial_0\int d^3x\, e^{-ix\cdot q}\overleftrightarrow{\partial}_0 A(x) \\
&= iZ_3^{-1/2}\varepsilon_{\lambda q}\cdot\int d^4x\,\partial_0\left(e^{-ix\cdot q}\partial_0 A(x) - A(x)\partial_0 e^{-ix\cdot q}\right) \\
&= iZ_3^{-1/2}\varepsilon_{\lambda q}\cdot\int d^4x\left(e^{-ix\cdot q}\partial_0^2 A(x) - A(x)\partial_0^2 e^{-ix\cdot q}\right) \\
&= iZ_3^{-1/2}\varepsilon_{\lambda q}\cdot\int d^4x\, e^{-ix\cdot q}\left(\partial_0^2 - \overleftarrow{\partial}_0^2\right)A(x) = iZ_3^{-1/2}\varepsilon_{\lambda q}\cdot\int d^4x\, e^{-ix\cdot q}\left(\partial_0^2 + q_0^2\right)A(x) \\
&= iZ_3^{-1/2}\varepsilon_{\lambda q}\cdot\int d^4x\, e^{-ix\cdot q}\left(\partial_0^2 - \overleftarrow{\nabla}^2\right)A(x) = iZ_3^{-1/2}\varepsilon_{\lambda q}\cdot\int d^4x\, e^{-ix\cdot q}\left(\partial_0^2 - \nabla^2\right)A(x) \\
&= iZ_3^{-1/2}\varepsilon_{\lambda q}\cdot\int d^4x\, e^{-ix\cdot q}\Box A(x),
\end{aligned}
$$

where we used integration by parts twice to get from $\overleftarrow{\nabla}^2$ to ∇^2. By Hermitian conjugation we find

$$a_{+,\lambda q} - a_{-,\lambda q} = -iZ^{-1/2}\varepsilon_{\lambda q} \cdot \int d^4x \ \ e^{iq\cdot x}\Box A(x).$$

7.5.7 Alternative Form of the LSZ Reduction Formula

We start with our "old" LSZ reduction formula, where we write the S-matrix element in terms of the ladder operators of the "in" and "out" fields, then insert our effective formulas for them given in the *Overview* in section 7.5 and finally apply time ordering:

$$\begin{aligned}S_{\beta\alpha} &= \langle\{q_i\}_+|\{p_i\}_-\rangle = \langle\Omega\big|a_{+,q_1}\cdots a^\dagger_{-,p_1}\cdots\big|\Omega\rangle \\ &= \int\prod_{i=1}^{n}\frac{i}{\sqrt{Z}}d^4x_i\ e^{iq_i\cdot x_i}(\Box_i + m^2)\prod_{j=1}^{m}\frac{i}{\sqrt{Z}}d^4y_j e^{-ip_j\cdot y_j}\big(\Box_j + m^2\big) \\ &\quad\cdot\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)|\Omega\rangle.\end{aligned}$$

We can now express the time ordered vacuum matrix element in terms of its Fourier transformed form, which we will, for now, call Γ:

$$\begin{aligned}&\Gamma(q_1,\dots,q_n,p_1\dots,p_m) \\ &\quad:= \int\prod_{a=1}^{n}d^4x_a\ e^{iq_a\cdot x_a}\prod_{b=1}^{m}d^4x_b\ e^{ip_b\cdot y_b}\,\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)|\Omega\rangle.\end{aligned}$$

The inverse relation reads (with k's instead of q's and k''s instead of p's)

$$\begin{aligned}&\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)|\Omega\rangle \\ &\quad= \int\prod_{a=1}^{n}d^4\bar{k}_a\ e^{-ik_a\cdot x_a}\prod_{b=1}^{m}d^4\bar{k}'_b\ e^{-ik'_b\cdot y_b}\,\Gamma(k_1,\dots,k_n,k'_1\dots,k'_m).\end{aligned}$$

If we plug in this inverse relation (and turn the integration variables $k'_b \to -k'_b$), we find

$$\begin{aligned}S_{\beta\alpha} &= \langle\{q_i\}_+|\{p_i\}_-\rangle = \langle\Omega\big|a_{+,q_1}\cdots a^\dagger_{-,p_1}\cdots\big|\Omega\rangle \\ &= \int\prod_{i=1}^{n}\frac{i}{\sqrt{Z}}d^4x_i\ e^{iq_i\cdot x_i}(\Box_i + m^2)\prod_{j=1}^{m}\frac{i}{\sqrt{Z}}d^4y_j e^{-ip_j\cdot y_j}\big(\Box_j + m^2\big) \\ &\qquad\cdot\int\prod_{a=1}^{n}d^4\bar{k}_a\ e^{-ik_a\cdot x_a}\prod_{b=1}^{m}d^4\bar{k}'_b\ e^{ik'_b\cdot y_b}\,\Gamma(k_1,\dots,k_n,-k'_1\dots,-k'_m) \\ &= \int\prod_{i=1}^{n}\frac{i}{\sqrt{Z}}d^4x_i\ e^{iq_i\cdot x_i}(\Box_i + m^2)\,d^4\bar{k}_i\ e^{-ik_i\cdot x_i}\prod_{j=1}^{m}\frac{i}{\sqrt{Z}}d^4y_j e^{-ip_j\cdot y_j}\big(\Box_j + m^2\big)d^4\bar{k}'_j\ e^{ik'_j\cdot y_j} \\ &\qquad\cdot\Gamma(k_1,\dots,k_n,-k'_1\dots,-k'_m) \\ &= \int\prod_{i=1}^{n}\frac{i}{\sqrt{Z}}d^4x_i\ d^4\bar{k}_i\ e^{i(q_i-k_i)\cdot x_i}\big(-k_i^2 + m^2\big)\prod_{j=1}^{m}\frac{i}{\sqrt{Z}}d^4y_j\ d^4\bar{k}'_j\ e^{-i(p_j-k'_j)\cdot y_j}\big(-k_j'^2 + m^2\big) \\ &\qquad\cdot\Gamma(k_1,\dots,k_n,-k'_1\dots,-k'_m) \\ &= \int\prod_{i=1}^{n}\frac{i}{\sqrt{Z}}d^4\bar{k}_i\ (2\pi)^4\delta(q_i - k_i)\big(-k_i^2 + m^2\big)\prod_{j=1}^{m}\frac{i}{\sqrt{Z}}d^4\bar{k}'_j\ (2\pi)^4\delta\big(p_j - k'_k\big)\big(-k_j'^2 + m^2\big) \\ &\qquad\cdot\Gamma(k_1,\dots,k_n,-k'_1\dots,-k'_m) \\ &= \prod_{i=1}^{n}\frac{1}{i\sqrt{Z}}\big(q_i^2 - m^2\big)\prod_{j=1}^{m}\frac{1}{i\sqrt{Z}}\big(p_j^2 - m^2\big)\cdot\Gamma(q_1,\dots,q_n,-p_1,\dots,-p_m).\end{aligned}$$

All integrals have now disappeared and we can put all the factors within the product signs on the other side of the equation. If we also express Γ again by the S-matrix element $S_{\beta\alpha} = \langle\{q_i\}_+|\{p_i\}_-\rangle$, we will get the desired equation given in section 7.5. Note that we changed the integration variable $p_i \to -p_i$ again (on the right-hand side).

7.7 Time-Evolution and Interaction Picture

7.7.1 Schrödinger Picture

In the Schrödinger picture, states evolve in time like

$$i\frac{d|\psi(t)\rangle_S}{dt} = H_S|\psi(t)\rangle_S,$$

while the operators O_S, like H_S, are independent of time. Schrödinger states evolve in time like

$$|\psi(t)\rangle_S = e^{-iH_S(t-t_0)}|\psi(t_0)\rangle_S,$$

which is consistent with the Schrödinger equation:

$$i\frac{d|\psi(t)\rangle_S}{dt} = i\frac{d}{dt}e^{-iH_S(t-t_0)}|\psi(t_0)\rangle_S = H_S e^{-iH_S(t-t_0)}|\psi(t_0)\rangle_S = H_S|\psi(t)\rangle_S.$$

Thus, the time-evolution operator reads $U(t,t_0) = e^{-iH_S(t-t_0)}$.

7.7.2 Heisenberg Picture

In the Heisenberg picture, the states are fixed and operators are time-dependent. We can construct a constant state $|\psi\rangle_H$ from $|\psi(t)\rangle_S$ by

$$|\psi\rangle_H \coloneqq |\psi(0)\rangle_S = U(0,t)|\psi(t)\rangle_S = e^{iHt}|\psi(t)\rangle_S.$$

If we define the operator in the Heisenberg picture as

$$O_H(t) \coloneqq U^{-1}(t,0)O_S U(t,0) = e^{iHt}O_S e^{-iHt},$$

the expectation values in the Schrödinger and Heisenberg picture are the same:

$${}_H\langle\psi|O_H(t)|\psi\rangle_H = {}_S\langle\psi(t)|e^{-iHt}e^{iHt}O_S e^{-iHt}e^{iHt}|\psi\rangle_S = {}_S\langle\psi(t)|O_S|\psi\rangle_S.$$

7.7.3 Interaction Picture

The interaction picture is a hybrid of the two. We split the Hamiltonian up as $H = H_0 + H_{\text{Int}}$ and we let H_0 govern the time-evolution of the operators and H_{Int} of the states. We define the states and operators in the interaction picture like

$$\begin{aligned}
&|\psi(t)\rangle_I \coloneqq e^{iH_0 t}|\psi(t)\rangle_S,\\
&O_I(t) \coloneqq e^{iH_0 t}O_S e^{-iH_0 t},
\end{aligned}$$

i.e. both are time-dependent. Obviously, we have again

$${}_I\langle\psi(t)|O_I(t)|\psi(t)\rangle_I = {}_S\langle\psi(t)|e^{-iH_0 t}e^{iH_0 t}O_S e^{-iH_0 t}e^{iH_0 t}|\psi\rangle_S = {}_S\langle\psi(t)|O_S|\psi\rangle_S.$$

Note that for a free theory, where $H_{\text{Int}} = 0$, we have $O_I(t) = O_H(t)$. If we plug in $|\psi(t)\rangle_S = e^{-iH_0 t}|\psi(t)\rangle_I$ into the Schrödinger equation, we find

$$\begin{aligned}
&i\frac{d}{dt}\left(e^{-iH_0 t}|\psi(t)\rangle_I\right) = H_0 e^{-iH_0 t}|\psi(t)\rangle_I + e^{-iH_0 t}i\frac{d}{dt}|\psi(t)\rangle_I \stackrel{!}{=} (H_0 + H_{\text{Int}})e^{-iH_0 t}|\psi(t)\rangle_I\\
\Leftrightarrow \quad & i\frac{d}{dt}|\psi(t)\rangle_I \stackrel{!}{=} e^{iH_0 t}H_{\text{Int}}e^{-iH_0 t}|\psi(t)\rangle_I = H_{\text{Int},I}(t)|\psi(t)\rangle_I.
\end{aligned}$$

Thus, only H_{Int} appears in the interaction picture Schrödinger equation and in this sense H_{Int} governs the time-evolution of the states.

We saw in the lecture *Quantum Mechanics II* that the time-evolution operator, which connects states in the interaction picture like $|\psi(t)\rangle_I = U_I(t,t_0)|\psi(t_0)\rangle_I$ can be given as

$$U_I(t,t_0) = \mathcal{T}\exp\left(-i\int_{t_0}^{t} dt'\, H_{\mathrm{Int},I}(t')\right).$$

7.8 Pictures in Quantum Fields Theory

7.8.1 From Heisenberg to Interaction Picture

φ are the fields we had all the time and they are to be understood in the Heisenberg picture. From the formulas of the quantum mechanics pictures, we would expect

$$\varphi = e^{iHt}\varphi_S e^{-iHt}, \quad \varphi_I = e^{iH_0 t}\varphi_S e^{-iH_0 t} \quad \Longrightarrow \quad \varphi_I = e^{iH_0 t}e^{-iHt}\varphi e^{iHt}e^{-iH_0 t}.$$

However, in our formula additional times t_0 appear. That is to the fact that the field in the Schrödinger picture can be understood as the Heisenberg field at some fixed time t_0 and the Heisenberg field is the time-evolved Schrödinger field:

$$\varphi(t,\vec{x}) = U^{-1}(t,t_0)\varphi_S(t_0,\vec{x})U(t,t_0) = e^{iH(t-t_0)}\varphi_S(t_0,\vec{x})e^{-iH(t-t_0)}.$$

This is a quite trivial generalization of the formula from (>7.7.2). In analogy, we now also have

$$\varphi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\varphi_S(t_0,\vec{x})e^{-iH_0(t-t_0)}$$

and we arrive at

$$\varphi_I(t,\vec{x}) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\varphi(t,\vec{x})e^{iH(t-t_0)}e^{-iH_0(t-t_0)}.$$

We now define $\widetilde{U}(t,t_0) := e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$ such that we can write

$$\varphi_I(t,\vec{x}) = \widetilde{U}(t,t_0)\varphi(t,\vec{x})\widetilde{U}^{-1}(t,t_0).$$

7.8.2 $\boldsymbol{\widetilde{U} = U_I}$

This operator $\widetilde{U}(t,t_0)$ turns now out to be the same as the time-evolution operator of the interaction picture $U_I(t,t_0)$, which can be developed as a Dyson series and written as a time-ordered exponential like in (>7.7.3). Let's prove this by taking the derivative:

$$\begin{aligned} i\frac{\partial}{\partial t}\widetilde{U}(t,t_0) &= e^{iH_0(t-t_0)}\underbrace{(H-H_0)}_{=H_{\mathrm{Int}}}e^{-iH(t-t_0)} = \underbrace{e^{iH_0(t-t_0)}H_{\mathrm{Int}}e^{-iH_0(t-t_0)}}_{=H_{\mathrm{Int},I}}\underbrace{e^{iH_0(t-t_0)}e^{-iH(t-t_0)}}_{=\widetilde{U}(t,t_0)} \\ &= H_{\mathrm{Int},I}\widetilde{U}(t,t_0). \end{aligned}$$

This is exactly the formula where one starts to derive the Dyson series: If we go back to (>7.7.3), where we found

$$i\frac{d}{dt}|\psi(t)\rangle_I = H_{\mathrm{Int},I}(t)|\psi(t)\rangle_I, \quad |\psi(t)\rangle_I = U_I(t,t_0)|\psi(t_0)\rangle_I,$$

we can plug in the latter into the former formula and find

$$i\frac{d}{dt}U_I(t,t_0)|\psi(t_0)\rangle_I = H_{\mathrm{Int},I}(t)U_I(t,t_0)|\psi(t_0)\rangle_I \quad \Leftrightarrow \quad i\frac{d}{dt}U_I(t,t_0) = H_{\mathrm{Int},I}(t)U_I(t,t_0).$$

Thus, we have indeed

$$\widetilde{U}(t,t_0) = U_I(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} = \mathcal{T}\exp\left(-i\int_{t_0}^{t} dt'\, H_{\mathrm{Int},I}(t')\right).$$

7.8.3 Picture of the S-Operators

Since U_I evolve interaction picture states in time, the S-operator, when given as $S = U_I$, should also be understood in the interaction picture, but since we know from (>7.7.3) and (>7.7.2) that

$$\langle\beta_\pm|S|\alpha_\pm\rangle = {}_I\langle\beta_\pm|S_I|\alpha_\pm\rangle_I,$$

where $\langle\beta_\pm|S|\alpha_\pm\rangle$ is in the Heisenberg picture, we do not need to put so much emphasis on the picture in which S is to be understood.

7.9 N-Point Functions

7.9.1 N-Point Function in Terms of "free" Fields

We saw that, for any *fixed* time t_0,

$$\varphi_I(t,\vec{x}) = U_I(t,t_0)\,\varphi(t,\vec{x})\,U_I^{-1}(t,t_0).$$

If we abbreviate $\varphi_1 \coloneqq \varphi(x_1)$, $\varphi_{I1} \coloneqq \varphi_I(x_1)$ and $\mathcal{U}_1 \coloneqq U_I(t_1,t_0)$, we can write

$$G(x_1,\dots,x_n) = \langle\Omega|\mathcal{T}\varphi_1\varphi_2\cdots\varphi_n|\Omega\rangle = \langle\Omega|\mathcal{T}\ \ \mathcal{U}_1^{-1}\varphi_{I1}\mathcal{U}_1\ \ \mathcal{U}_2^{-1}\varphi_{I2}\mathcal{U}_2\ \ \cdots\ \ \mathcal{U}_n^{-1}\varphi_{In}\mathcal{U}_n|\Omega\rangle.$$

Now we find quite intuitively, using $U_I^{-1}(t,t_0) = U_I(t_0,t)$,

$$\mathcal{U}_i\mathcal{U}_j^{-1} = U_I(t_i,t_0)U_I^{-1}(t_j,t_0) = U_I(t_i,t_0)U_I(t_0,t_j) = U_I(t_i,t_j) =: \mathcal{U}_{ij}.$$

The step $U_I(t_i,t_0)U_I(t_0,t_j) = U_I(t_i,t_j)$ is obvious from the exponential representation of U_I. We now can write

$$\begin{aligned}
G(x_1,\dots,x_n) &= \langle\Omega|\mathcal{T}\ \ \mathcal{U}_1^{-1}\ \varphi_{I1}\ \mathcal{U}_{12}\ \varphi_{I2}\ \mathcal{U}_{23}\ \cdots\ \mathcal{U}_{n-1,n}\ \varphi_{In}\ \mathcal{U}_n|\Omega\rangle \\
&= \langle\Omega|\mathcal{T}\ \mathcal{U}_\infty^{-1}\mathcal{U}_\infty\ \mathcal{U}_1^{-1}\ \varphi_{I1}\ \mathcal{U}_{12}\ \varphi_{I2}\ \mathcal{U}_{23}\ \cdots\ \mathcal{U}_{n-1,n}\ \varphi_{In}\ \mathcal{U}_n\ \mathcal{U}_{-\infty}^{-1}\mathcal{U}_{-\infty}|\Omega\rangle \\
&= \langle\Omega|\mathcal{U}_\infty^{-1}\ \mathcal{T}\ \mathcal{U}_{\infty 1}\ \varphi_{I1}\ \mathcal{U}_{12}\ \varphi_{I2}\ \mathcal{U}_{23}\ \cdots\ \mathcal{U}_{n-1,n}\ \varphi_{In}\ \mathcal{U}_{n,-\infty}\ \mathcal{U}_{-\infty}|\Omega\rangle,
\end{aligned}$$

where obviously $\mathcal{U}_\infty \coloneqq U_I(\infty,t_0)$ and $\mathcal{U}_{\infty 1} \coloneqq U_I(\infty,t_1)$. Obviously, $\mathcal{U}_\infty^{-1}$ carries no time dependence, which is why we were able to pull it to the left-hand side of $\mathcal{T}$. Since we have our time-ordering operator there, we can commute the $\mathcal{U}$'s with the fields and write

$$G(x_1,\dots,x_n) = \langle\Omega|\mathcal{U}_\infty^{-1}\ \ \mathcal{T}\ \ \varphi_{I1}\varphi_{I2}\cdots\varphi_{In}\ \ \mathcal{U}_{\infty 1}\mathcal{U}_{12}\mathcal{U}_{23}\cdots\mathcal{U}_{n-1,n}\mathcal{U}_{n,-\infty}\ \ \mathcal{U}_{-\infty}|\Omega\rangle.$$

Using the integral definition for U_I it is easy to see that $\mathcal{U}_{ij}\mathcal{U}_{jk} = \mathcal{U}_{ik}$ for $t_i > t_j > t_k$. We can therefore write

$$\begin{aligned}
G(x_1,\dots,x_n) &= \langle\Omega|\mathcal{U}_\infty^{-1}\ \ \mathcal{T}\ \ \varphi_{I1}\varphi_{I2}\cdots\varphi_{In}\ \ \mathcal{U}_{\infty,-\infty}\ \ \mathcal{U}_{-\infty}|\Omega\rangle \\
&= \langle\Omega|\mathcal{U}_\infty^{-1}\ \ \mathcal{T}\ \ \varphi_{I1}\varphi_{I2}\cdots\varphi_{In}\ \ \exp\left(-i\int_{-\infty}^{\infty}dt'\,H_{\mathrm{Int},I}(t')\right)\ \ \mathcal{U}_{-\infty}|\Omega\rangle.
\end{aligned}$$

Now we need to deal with the $\mathcal{U}$-operators acting on the vacuum states $|\Omega\rangle$. We will use $H|\Omega\rangle = 0$ and thus $e^{iHt}|\Omega\rangle = |\Omega\rangle$. Finally, we will insert a full spectrum of H_0-eigenstates:

$$\begin{aligned}
\mathcal{U}_{\pm\infty}|\Omega\rangle = U_I(\pm\infty,t_0)|\Omega\rangle &= \lim_{t\to\pm\infty} e^{iH_0(t-t_0)}e^{-iH(t-t_0)}|\Omega\rangle = \lim_{t\to\pm\infty} e^{iH_0(t-t_0)}|\Omega\rangle \\
&= \lim_{t\to\pm\infty}\sum_n e^{iH_0(t-t_0)}|n\rangle\langle n|\Omega\rangle = \lim_{t\to\pm\infty}\sum_n e^{iE_n(t-t_0)}|n\rangle\langle n|\Omega\rangle \\
&= |0\rangle\langle 0|\Omega\rangle + \lim_{t\to\pm\infty}\sum_{n\neq 0} e^{iE_n(t-t_0)}|n\rangle\langle n|\Omega\rangle.
\end{aligned}$$

Actually, we have continuous states, i.e. the sum is an integral. The Riemann-Lebesgue lemma tells us that

$$\lim_{\xi\to\infty}\int_a^b dx\, f(x)e^{\pm i\xi x} = 0.$$

We could also see this when instead of $t \to -\infty$ taking the limit $t \to \pm\infty(1 \pm i\epsilon)$ for an arbitrarily small, but fixed ϵ. Anyway, we are left with

$$\mathcal{U}_{\pm\infty}|\Omega\rangle = |0\rangle\langle 0|\Omega\rangle \quad \Leftrightarrow \quad |\Omega\rangle = \mathcal{U}_{\pm\infty}^{-1}|0\rangle\langle 0|\Omega\rangle$$

Recall at this point that $\mathcal{U}^{-1} = \mathcal{U}^{\dagger}$. Now, we can write

$$G(x_1, \dots, x_n) = \langle\Omega|0\rangle\langle 0\big|\mathcal{T} \quad \varphi_{I1}\varphi_{I2}\cdots\varphi_{In} \quad \exp\left(-i\int_{-\infty}^{\infty} dt'\, H_{\text{Int},I}(t')\right)\big|0\rangle\langle 0|\Omega\rangle.$$

We are now left with finding $\langle\Omega|0\rangle$. We use the normalization[1] $\langle\Omega|\Omega\rangle = 1$

$$\begin{aligned} 1 = \langle\Omega|\Omega\rangle &= \langle\Omega|0\rangle\langle 0|\mathcal{U}_{\infty}\mathcal{U}_{-\infty}^{-1}|0\rangle\langle 0|\Omega\rangle = \langle\Omega|0\rangle\langle 0|\Omega\rangle\langle 0|U_I(\infty, t_0)U_I^{-1}(-\infty, t_0)|0\rangle \\ &= \langle\Omega|0\rangle\langle 0|\Omega\rangle\langle 0|U_I(\infty, t_0)U_I(t_0, -\infty)|0\rangle = \langle\Omega|0\rangle\langle 0|\Omega\rangle\langle 0|U_I(\infty, -\infty)|0\rangle \\ \Leftrightarrow \quad \langle\Omega|0\rangle\langle 0|\Omega\rangle &= \frac{1}{\langle 0\big|\mathcal{T}\exp\left(i\int_{-\infty}^{\infty} d^4x\, \mathcal{L}_{\text{Int},I}\right)\big|0\rangle}, \end{aligned}$$

where we used

$$S = U_I(\infty, -\infty) = \mathcal{T}\exp\left(i\int_{-\infty}^{\infty} d^4x\, \mathcal{L}_{\text{Int},I}\right).$$

7.9.2 Interaction Picture Fields are Free Heisenberg Picture Fields

We saw that the connection between Schrödinger, Heisenberg and Interaction operators reads

$$O_H(t) = e^{iHt}O_S e^{-iHt}, \quad O_I(t) = e^{iH_0T}O_S e^{-iH_0 t}.$$

In case of *field* operators, we saw in (>7.8.1) that this is easily generalized:

$$\varphi_H(t, \vec{x}) = e^{iH(t-t_0)}\varphi_S(t_0, \vec{x})e^{-iH(t-t_0)}, \quad \varphi_I(t, \vec{x}) = e^{iH_0(t-t_0)}\varphi_S(t_0, \vec{x})e^{-iH_0(t-t_0)}.$$

Let's pick t_0 to be a time far from the scattering, such that $\varphi_S(t_0, \vec{x})$ is a *free* Schrödinger field. To get the *free* Heisenberg field, we have to use the free Hamiltonian H_0 instead of the total H in the exponent – and from there it is obvious that $\varphi_{H,\text{free}} = \varphi_I$.

7.10 Wick's Theorem

7.10.1 Proof of Wick's Theorem

Any free Heisenberg field (or equivalently, interaction picture field of an interacting theory) φ can be split up into $\varphi^{\pm}$, where φ^{+} contains the $a^{\dagger}$-term of the expansion and φ^{-} the a-term. For example, for fermions this is

$$\begin{aligned} \psi &= \int d\tilde{p}\, \left(b_{\alpha p}^{\dagger} v_p^{\alpha} e^{ip\cdot x} + a_{\alpha p} u_p^{\alpha} e^{-ip\cdot x}\right) \\ \Rightarrow \quad \psi^{+} &= \int d\tilde{p}\, b_{\alpha p}^{\dagger} v_p^{\alpha} e^{ip\cdot x}, \quad \psi^{-} = \int d\tilde{p}\, a_{\alpha p} u_p^{\alpha} e^{-ip\cdot x}. \end{aligned}$$

This decomposition is useful, because $\langle 0|\varphi^{+} = 0$ and $\varphi^{-}|0\rangle = 0$.

N-POINT FUNCTION FOR ODD N:

[1] I assume that if we take the normalization $\langle\Omega|\Omega\rangle = 1$, $\langle 0|0\rangle = 1$ is not true anymore. Otherwise we get some contradictions, here. However, all textbooks assume $\langle 0|0\rangle = 1$ later again, for example in Wick's theorem. Apparently, you can switch normalizations as needed …

Consider a 3-point function, using $\varphi_i := \varphi(x_i)$. Without loss of generality we assume $t_1 > t_2 > t_3$, such that we do not need to re-order the fields when dropping the $\mathcal{T}$ (otherwise we re-order it correspondingly and the proof goes through in the same way):

$$\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3|0\rangle = \langle 0|\varphi_1\varphi_2\varphi_3|0\rangle = \langle 0|\varphi_1\varphi_2(\varphi_3^+ + \varphi_3^-)|0\rangle = \langle 0|\varphi_1\varphi_2\varphi_3^+|0\rangle.$$

Here, we used already $\varphi^-|0\rangle = 0$. Now we use the symbol

$$[A,B]_\pm := AB \pm BA \quad \Leftrightarrow \quad AB = [A,B]_\pm \mp BA.$$

for the commutator and anticommutator respectively.[1] We will now push through φ_3^+ to the left and thereby catch commutators (for boson fields) or anticommutators (for fermion fields). The commutators/anticommutators for boson/fermion fields are always complex numbers (no operators at least, not few of them are zero) and thereby can be pulled out of the expectation value:

$$\begin{aligned}\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3|0\rangle &= \langle 0|\varphi_1([\varphi_2,\varphi_3^+]_\pm \mp \varphi_3^+\varphi_2)|0\rangle = \langle 0|\varphi_1|0\rangle[\varphi_2,\varphi_3^+]_\pm \mp \langle 0|\varphi_1\varphi_3^+\varphi_2|0\rangle \\ &= \langle 0|\varphi_1|0\rangle[\varphi_2,\varphi_3^+]_\pm \mp \langle 0|([\varphi_1,\varphi_3^+]_\pm \mp \varphi_3^+\varphi_1^-)\varphi_2|0\rangle \\ &= \underbrace{\langle 0|\varphi_1|0\rangle}_{=0}[\varphi_2,\varphi_3^+]_\pm \mp \underbrace{\langle 0|\varphi_2|0\rangle}_{=0}[\varphi_1,\varphi_3^+]_\pm + \underbrace{\langle 0|\varphi_3^+}_{=0}\varphi_1\varphi_2|0\rangle = 0.\end{aligned}$$

The crucial point is that the vacuum expectation value of a single field is always zero, since

$$\langle 0|\varphi|0\rangle = \underbrace{\langle 0|(\varphi^+}_{=0} + \underbrace{\varphi^-)|0\rangle}_{=0} = 0.$$

It is quite obvious now that *all* n-point functions for odd n vanish: When we push through the φ_n^+ field from the right-hand side, each step produces an additional commutator-/anticommutator-term. Since the commutator/anticommutator is pulled out of the expectation value, the number of fields within the expectation value is reduced by two. If it was odd in the beginning, the number is also odd after reduction by two. We now have the vacuum expectation value of $n-2$ fields, which we can treat in the same way as the one for n fields: We can reduce it further until we reach the expectation value of a single field and this vanishes. What is left is the one term without commutators/anticommutators and with all n fields. But here the φ_n^+ field is now placed on the very left, where it vanishes when acting on $\langle 0|$.

N-POINT FUNCTION FOR EVEN N:

Let's consider a 4-point function and treat it in the same way as the 3-point function before. Again, we assume without loss of generality $t_1 > t_2 > t_3 > t_4$ and find

$$\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = \langle 0|\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = \langle 0|\varphi_1\varphi_2\varphi_3\varphi_4^+|0\rangle.$$

Now we push through the φ_4^+ to the left. We have to push it past three fields, thus we get three terms with commutators/anticommutators:

$$\begin{aligned}&\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle \\ &\quad = \langle 0|\varphi_1\varphi_2|0\rangle[\varphi_3,\varphi_4^+]_\pm \mp \langle 0|\varphi_1\varphi_3|0\rangle[\varphi_2,\varphi_4^+]_\pm + \langle 0|\varphi_2\varphi_3|0\rangle[\varphi_1,\varphi_4^+]_\pm \\ &\quad\quad \mp \langle 0|\varphi_4^+\varphi_1\varphi_2\varphi_3|0\rangle.\end{aligned}$$

The last term vanishes due to $\langle 0|\varphi_4^+ = 0$. We now use the following trick: We were able to pull out the commutators/anticommutators out of the vacuum expectation values. Of course, we can also push them back in and use once again $\langle 0|\varphi_4^+ = 0$:

$$[\varphi_3,\varphi_4^+]_\pm = \langle 0|[\varphi_3,\varphi_4^+]_\pm|0\rangle = \langle 0|\varphi_3\varphi_4^+ \pm \varphi_4^+\varphi_3|0\rangle = \langle 0|\varphi_3\varphi_4^+|0\rangle = \langle 0|\varphi_3\varphi_4|0\rangle.$$

Thus, we are left with

[1] The confusingly redundant notation $[A,B]_- := [A,B]$ and $[A,B]_+ := \{A,B\}$ is only used in this section to handle bosons and fermions at the same time.

$$\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = \langle 0|\varphi_1\varphi_2|0\rangle\langle 0|\varphi_3\varphi_4|0\rangle \mp \langle 0|\varphi_1\varphi_3|0\rangle\langle 0|\varphi_2\varphi_4|0\rangle + \langle 0|\varphi_1\varphi_4|0\rangle\langle 0|\varphi_2\varphi_3|0\rangle.$$

Note that the upper sign holds for fermion fields and the lower sign for boson fields. We observe that any odd permutation of the field order gives a minus sign for fermions (+1234, −1324, +1423).[1]

We did this all with the assumptions $t_1 > t_2 > t_3 > t_4$. If we get rid of that restriction, we obviously get all possible combinations *time-ordered* 2-point functions:

$$\begin{aligned}&\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle\\ &\quad = \pm\langle 0|\mathcal{T}\varphi_1\varphi_2|0\rangle\langle 0|\mathcal{T}\varphi_3\varphi_4|0\rangle \pm \langle 0|\mathcal{T}\varphi_1\varphi_3|0\rangle\langle 0|\mathcal{T}\varphi_2\varphi_4|0\rangle \pm \langle 0|\mathcal{T}\varphi_1\varphi_4|0\rangle\langle 0|\mathcal{T}\varphi_2\varphi_3|0\rangle.\end{aligned}$$

However, which of the terms carries the minus sign (in case on fermions), does depend on the order of the times t_i. For example, for $t_2 > t_1 > t_3 > t_4$ we have $\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = \langle 0|\varphi_2\varphi_1\varphi_3\varphi_4|0\rangle$, so we start with the order 2134. 1324 is now an even permutation and gets a plus sign, whereas 1423 is the odd permutation and ends up with a minus sign.

We know that 2-point function are just Feynman propagators. Using $D_{ij} \coloneqq D_F(x_i - x_j)$ we can write

$$\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = \pm D_{12}D_{34} \pm D_{13}D_{24} \pm D_{14}D_{23}.$$

7.10.2 Different Types of Fields

So far, we always took the commutator/anticommutator to be non-zero. What happens for different types of particles, for example photons and fermions, for which we know the commutator vanishes? Take a look at our formula from 7.10.1

$$\begin{aligned}&\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle\\ &\quad = \langle 0|\varphi_1\varphi_2|0\rangle[\varphi_3,\varphi_4^+]_\pm \mp \langle 0|\varphi_1\varphi_3|0\rangle[\varphi_2,\varphi_4^+]_\pm + \langle 0|\varphi_2\varphi_3|0\rangle[\varphi_1,\varphi_4^+]_\pm\\ &\qquad \mp \underbrace{\langle 0|\varphi_4^+}_{=0}\varphi_1\varphi_2\varphi_3|0\rangle.\end{aligned}$$

Let's assume the fields 1 and 3 as well as 2 and 4 are the same. Then we get

$$\langle 0|\mathcal{T}\varphi_1\tilde{\varphi}_2\varphi_3\tilde{\varphi}_4|0\rangle = \mp\langle 0|\varphi_1\varphi_3|0\rangle[\tilde{\varphi}_2,\tilde{\varphi}_4^+]_\pm = \mp\langle 0|\mathcal{T}\varphi_1\varphi_3|0\rangle\langle 0|\mathcal{T}\tilde{\varphi}_2\tilde{\varphi}_4|0\rangle.$$

Thus, we can always split up the n-point functions in n'-point functions containing only fields of one type each. For example, we find for photon fields A and fermion fields ψ

$$\begin{aligned}\langle 0|\mathcal{T}A_1A_2\psi_3\psi_4A_5A_6|0\rangle &= \langle 0|\mathcal{T}A_1A_2A_5A_6|0\rangle\langle 0|\mathcal{T}\psi_3\psi_4|0\rangle = \left(\hat{D}_{12}\hat{D}_{56} + \hat{D}_{15}\hat{D}_{26} + \hat{D}_{25}\hat{D}_{16}\right)\tilde{D}_{34}\\ &= \hat{D}_{12}\hat{D}_{56}\tilde{D}_{34} + \hat{D}_{15}\hat{D}_{26}\tilde{D}_{34} + \hat{D}_{25}\hat{D}_{16}\tilde{D}_{34}.\end{aligned}$$

[1] Note that this is consistent with the fact that $\langle 0|\varphi_1\varphi_2|0\rangle\langle 0|\varphi_3\varphi_4|0\rangle = \langle 0|\varphi_3\varphi_4|0\rangle\langle 0|\varphi_1\varphi_2|0\rangle$ since it takes an even number of permutations to get from 1234 to 3412.

8 Feynman Diagrams and Rules

8.1 ϕ⁴-Theory

8.1.1 Feynman Rules in Momentum Space

Let's consider the connected diagram only, for which we arrived, in first order, at the expression

$$\mathcal{S} = i^2 \int d^4x_1\, d^4x_2\, e^{-ip_2\cdot x_2} e^{ip_1\cdot x_1}\ (\Box_2 + m^2)(\Box_1 + m^2) \\ \left(-\frac{12i\lambda}{4!\,\mathcal{G}_D}\int d^4z\, D_F(x_2 - z) D_F(x_1 - z) D_F(z - z)\right).$$

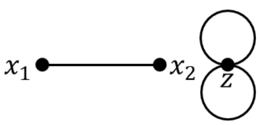
If we plug in the integral formulas for the propagators, we find

$$\begin{aligned}
\mathcal{S} &= -\frac{12i^3\lambda}{4!\,\mathcal{G}_D}\int d^4x_1\, d^4x_2\, d^4z\, d^4\bar{q}_1\, d^4\bar{q}_2\, d^4\bar{q}_3\, e^{-ip_2\cdot x_2} e^{ip_1\cdot x_1}\ (\Box_2 + m^2)(\Box_1 + m^2) \\
&\qquad \frac{i}{q_1^2 - m^2 + i\epsilon} e^{-iq_1\cdot(x_2 - z)} \frac{i}{q_2^2 - m^2 + i\epsilon} e^{-iq_2\cdot(x_1 - z)} \frac{i}{q_3^2 - m^2 + i\epsilon} e^{-iq_3\cdot(z - z)} \\
&= -\frac{12i^3\lambda}{4!\,\mathcal{G}_D}\int d^4x_1\, d^4x_2\, d^4z\, d^4\bar{q}_1\, d^4\bar{q}_2\, d^4\bar{q}_3\, e^{-ip_2\cdot x_2} e^{ip_1\cdot x_1} \\
&\qquad \frac{i(-q_1^2 + m^2)}{q_1^2 - m^2 + i\epsilon} e^{-iq_1\cdot(x_2 - z)} \frac{i(-q_2^2 + m^2)}{q_2^2 - m^2 + i\epsilon} e^{-iq_2\cdot(x_1 - z)} \frac{i}{q_3^2 - m^2 + i\epsilon} e^{-iq_3\cdot(z - z)} \\
&= -\frac{12i^3\lambda}{4!\,\mathcal{G}_D}\int d^4x_1\, d^4x_2\, d^4z\, d^4\bar{q}_1\, d^4\bar{q}_2\, d^4\bar{q}_3\, e^{-p_2\cdot x_2} e^{p_1\cdot x_1} \\
&\qquad ie^{-iq_1\cdot(x_2 - z)} ie^{-iq_2\cdot(x_1 - z)} \frac{ie^{-iq_3\cdot(z - z)}}{q_3^2 - m^2 + i\epsilon} \\
&= -(2\pi)^8 \frac{12i^3\lambda}{4!\,\mathcal{G}_D}\int d^4z\, d^4\bar{q}_1\, d^4\bar{q}_2\, d^4\bar{q}_3\, \delta(p_1 - q_2)\delta(-p_2 - q_1)\, ie^{iq_1\cdot z} ie^{iq_2\cdot z} \frac{ie^{-iq_3\cdot(z - z)}}{q_3^2 - m^2 + i\epsilon} \\
&= -\frac{12i^3\lambda}{4!\,\mathcal{G}_D}\int d^4z\, d^4\bar{q}_3\, ie^{-ip_2\cdot z} ie^{ip_1\cdot z} \frac{ie^{-iq_3\cdot(z - z)}}{q_3^2 - m^2 + i\epsilon} \\
&= -\frac{12i^6\lambda}{4!\,\mathcal{G}_D}\int d^4z\, d^4\bar{q}_3\, e^{-i(p_2 - p_1 + q_3 - q_3)\cdot z} \frac{1}{q_3^2 - m^2 + i\epsilon} \\
&= -\frac{12i^6\lambda}{4!\,\mathcal{G}_D}\int d^4\bar{q}_3\, \frac{(2\pi)^4\delta(p_2 - p_1 + q_3 - q_3)}{q_3^2 - m^2 + i\epsilon}.
\end{aligned}$$

The factor $\mathcal{G}_D$ aside, this is exactly what we got by our Feynman rules. One may also note that this q_3-integral is divergent. This is typical for diagrams with closed loops. It will later take us a great deal of effort to deal with these divergences.

8.1.2 Disconnected Parts of Diagrams

In general, one Feynman diagram as we defined it so far, can have one or more disconnected parts, which are closed in the sense that they do not have any incoming or outgoing particles. We already met one of those: In the diagram



the part of the z vertex is disconnected from the part with incoming/outgoing particles. Let $\{V_i\}$ be the set of all possible disconnected parts of ϕ^4-theory. Suppose a given diagram has n_i pieces of the form V_i for each i in addition to the connected piece. We see at the formula for $\mathcal{G}_N^1$ in terms of the propagators D_F that disconnected parts of a diagram will factor out (recall that $\mathcal{G}_N^1$ contains not only one but several *diagrams*).[1] The value of a given diagram W_d can then be given as

$$W_d = \mathcal{V}_d \prod_i \frac{1}{n_i(d)!} V_i^{n_i(d)},$$

where $\mathcal{V}_d$ is the value of its connected part. Of course, the number n_i of a disconnected part V_i depends on the diagram d. The factor $1/n_i!$ is a symmetry factor coming from the interchanging of the n_i copies of V_i. The matrix element contains now the sum of *all diagrams*:

$$\mathcal{S} \sim \sum_d W_d = \sum_d \mathcal{V}_d \prod_i \frac{1}{n_i(d)!} V_i^{n_i(d)}.$$

Of course, there are many diagrams, which have the same connected part $\mathcal{V}_d$ and differ only in the disconnected parts. For such a group of diagrams with the same connected part $\mathcal{V}_d$ we can factor out this connected part. What then remains "in the brackets" is the sum of all possible combinations of all possible disconnected parts, i.e. the sum

$$\sum_{\{n_i\}} \prod_i \frac{1}{n_i!} V_i^{n_i},$$

where $\{n_i\} = \{n_1, n_2, n_3, \cdots\}$. That is, we sum over $\{0,0,0,\dots\}$ and $\{1,0,0,\dots\}$ and also $\{256,12,34,\dots\}$ and so on. But this sum is *the same* for *any* group with the same connected part $\mathcal{V}_d$. Thus, we can in turn factor out this sum over $\{n_i\}$ and what remains now "in the brackets" is the sum over all *different* connected parts $\mathcal{V}_d$, which we will call $\sum$ connected:

$$\mathcal{S} \sim \sum_{\{n_i\}} \prod_i \frac{1}{n_i!} V_i^{n_i} \cdot \sum_{\substack{\text{diagramms } d \\ \text{with different } \mathcal{V}_d}} \mathcal{V}_d = \sum_{\{n_i\}} \prod_i \frac{1}{n_i!} V_i^{n_i} \cdot \sum \text{connected}.$$

It is now possible to write the sum over $\{n_i\}$ as an exponential:

$$\sum_{\{n_i\}} \prod_i \frac{1}{n_i!} V_i^{n_i} = \sum_{n_1} \sum_{n_2} \sum_{n_3} \cdots \prod_i \frac{1}{n_i!} V_i^{n_i} = \prod_i \sum_{n_i} \frac{1}{n_i!} V_i^{n_i} = \prod_i \exp V_i = \exp \sum_i V_i.$$

What we now see is that the sum of all diagrams is equal to the sum of all connected parts times the exponential of the sum of all disconnected parts. If we now take a look at

$$\mathcal{G}_D = \left\langle 0 \middle| \mathcal{T} \exp\left(i \int d^4z\, \mathcal{L}_{\text{Int},I} \right) \middle| 0 \right\rangle,$$

and expand it in the same way as $\mathcal{G}_N$, we will just get diagrams containing disconnected parts only. That s, we will get

$$\mathcal{G}_D = \sum_{\{n_i\}} \prod_i \frac{1}{n_i!} V_i^{n_i} = \exp \sum_i V_i,$$

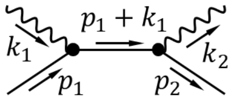
which cancels the disconnected parts of $\mathcal{G}_N$.

[1] To get this straight: The matrix element of a physical (scattering) process can be described as a series of (infinitely) many Feynman *diagrams* and each *diagram* may have one or more disconnected *parts*. The *parts* of a given *diagram* factor out of this *diagram* but certainly not of the *series of all diagrams*.

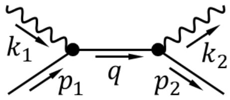
8.2 The Feynman Rules of QED

8.2.1 Energy Momentum Conservation at each Vertex

Instead of labeling all internal momenta independently with q_i, including the four-momentum conservation δ-function for each vertex and integration over $d^4\bar{q}_i$, one can already label internal momenta by expressions of external momenta, such that the four-momentum is conserved at each vertex, for example



instead of



Then one can save time because one does not have to do the integrals over internal momenta.

8.2.2 S-Matrix, T-Matrix, Amplitude

The structure of the S-matrix is $S = 1 + iT$, where 1 is the identity operator: Even in an interacting theory, particles may simply miss each other and don't interact. In practice, we do never draw such Feynman diagrams, where particles don't interact, because we don't care about them. Thus, when drawing (interacting) Feynman diagrams, we only consider the iT part of S, where particles do interact. Moreover, when using Feynman rules, we will – after the integration over internal momenta – *always* end up with a δ-function of total momentum conservation.

Let's consider an arbitrary S-matrix element with incoming momenta $\{p_i\} \coloneqq \{p_1, p_2, \dots\}$ and outgoing momenta $\{p_f\} \coloneqq \{p_1', p_2', \dots\}$:

$$\langle\{p_f\}|S|\{p_i\}\rangle = \langle\{p_f\}|\{p_i\}\rangle + \langle\{p_f\}|iT|\{p_i\}\rangle \mathrel{\widehat{=}} \langle\{p_f\}|iT|\{p_i\}\rangle = (2\pi)^4\delta(p_i - p_f)\cdot i\mathcal{M}(\{p_f\},\{p_i\}).$$

We dropped $\langle\{p_f\}|\{p_i\}\rangle$ due to lack of interest. Inside the δ-function, by p_i and p_f we obviously mean $p_i \coloneqq \sum_n p_n$ *and* $p_f \coloneqq \sum_n p_n'$. The usual procedure is now that the result of the Feynman rules is “”per definition” $i\mathcal{M}$. Since the Feynman rules also produce a δ-functon $\delta(p_i - p_f)$, we add another Feynman rule, according to which, we have to drop the total momentum conservation δ-function (together with a factor of $(2\pi)^4$) in the end. *Those* Feynman rules then will give us $i\mathcal{M}$.

$\mathcal{M}$ is called “amplitude”, but often also simply “matrix element”.

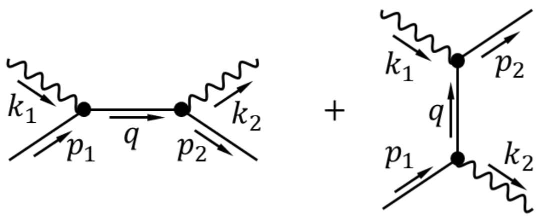
8.3 Compton Scattering

8.3.1 S-Matrix Element using Feynman Rules

Let's consider Compton scattering, where a photon with momentum k_1 and an electron with momentum p_1 scatter and leave with a momentum k_2 and p_2 respectively:

$$\gamma(k_1) + e^-(p_1) \to \gamma(k_2) + e^-(p_2).$$

Up to second order (two vertices), there are two Feynman diagrams for this process:



Using the QED Feynman rules for each of the two diagrams, we get the expression

$$
\begin{aligned}
&Z_2 Z_3 \int d^4\bar{q}\, \bar{u}_{p_2} \varepsilon_{k_2\mu}\, ig\gamma^\mu \frac{i}{\not{q} - m + i\epsilon}\, ig\gamma^\nu\, \varepsilon_{k_1\nu} u_{p_1} (2\pi)^8 \delta(p_1 + k_1 - q)\delta(q - p_2 - k_2) \\
&\quad + Z_2 Z_3 \int d^4\bar{q}\, \bar{u}_{p_2} \varepsilon_{k_1\mu}\, ig\gamma^\mu\, \frac{i}{\not{q} - m + i\epsilon}\, ig\gamma^\nu\, \varepsilon_{k_2\nu} u_{p_1}\, (2\pi)^8 \delta(q + k_1 - p_2)\delta(p_1 - k_2 - q) \\
&\quad = -Z_2 Z_3 g^2 (2\pi)^4 \delta(p_1 + k_1 - p_2 - k_2) \\
&\qquad\qquad\qquad \bar{u}_{p_2} \left(\not{\varepsilon}_{k_2} \frac{i}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \not{\varepsilon}_{k_1} + \not{\varepsilon}_{k_1} \frac{i}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \not{\varepsilon}_{k_2} \right) u_{p_1}.
\end{aligned}
$$

Applying the last Feynman rule, we drop the factor $(2\pi)^4\delta(\cdots)$ to arrive at $i\mathcal{M}$:

$$
i\mathcal{M} = -Z_2 Z_3 g^2 \bar{u}_{p_2} \left(\not{\varepsilon}_{k_2} \frac{i}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \not{\varepsilon}_{k_1} + \not{\varepsilon}_{k_1} \frac{i}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \not{\varepsilon}_{k_2} \right) u_{p_1}.
$$

Some hints about the Feynman rules:

- $i\mathcal{M}$ is a scalar, not a matrix. Thus, the order of the spinors and the γ-matrices must always be conjugated spinor, γ-matrices, spinor. For example $\bar{u}_{p_2}\gamma^\mu u_{p_1}$.
- Note that also the propagator $i/(\not{q} - m + i\epsilon) = i(\not{p} + m)/(q^2 - m^2 + i\epsilon)$ is a matrix and its position w.r.t γ-matrices and spinors is critical. If the corresponding propagator line in the diagram sits *between* two vertices, then the propagator $i/(\not{q} - m + i\epsilon)$ must be placed in *between* the γ-matrices of those vertices.
- The polarization vector ε of the photon must be contracted with the γ-matrix of the vertex the corresponding photon line is attached to.
- The δ-function, which ensure 4-momentum conservation at each vertex, can always be constructed in the following way:
  $\delta(\text{4-momentum conservation}) = \delta(\sum \text{incoming momenta} - \sum \text{outgoing momenta})$
  In this sense here, "incoming" and "outgoing" *does not* refer to "into/out of the diagram" as it is the case for "incoming/outgoing particles/antiparticles/photons" but to "into/out of the vertex".
- After performing the integrals over internal momenta, we will *always* be left with *one* δ-function, which ensured 4-momentum conservation *of the whole process*. Obviously, it must only contain *external* momenta.

8.3.2 Rigorous Calculation of Compton Scattering

What we want to do now is to explicitly calculate the corresponding S-matrix element $\mathcal{S} := \langle k_1, p_1 | k_2, p_2 \rangle$ for the theory of QED, that is to say for an interaction Lagrangian of the form

$$
\mathcal{L}_{\text{Int}} = g\bar{\psi}(z)\gamma^\sigma\psi(z)A_\sigma(z)
$$

(g is the elementary charge e, but e is also the Euler's number, so we will use g here).

LSZ REDUCTION:

Obviously, we have an interacting theory, therefore we need to write our matrix element with the interacting vacuum $|\Omega\rangle$ and time-dependent ladder operators, as discussed for the LSZ reduction:

$$
\mathcal{S} = \langle \{k_2, p_2\}_+ | \{k_1, p_1\}_- \rangle = \left\langle \Omega \middle| a_{+,\lambda_2 k_2} a_{+,\alpha_2 p_2} a^\dagger_{-,\lambda_1 k_1} a^\dagger_{-,\alpha_1 p_1} \middle| \Omega \right\rangle.
$$

Note that $\mathcal{M}$ actually not only depends on the momenta but also on the polarizations $\lambda_{1,2}$ and the spins $\alpha_{1,2}$. From now on, we will not explicitly note down those dependencies anymore. We now plug in the LSZ reduction formulas for the ladder operators (we neglect the self-energies due to 7.6):

$$
\begin{aligned}
\mathcal{S} = i(-i)^3 \int & d^4y_2\, d^4x_2\, d^4y_1\, d^4x_1 \quad e^{ik_2\cdot y_2} e^{ip_2\cdot x_2} e^{-ik_1\cdot y_1} \quad \varepsilon_{k_2\mu}\varepsilon_{k_1\nu} \\
& \bar{u}_{p_2} \quad \Box_{y_2}\Box_{y_1}\left(i\not{\partial}_{x_2} - m\right) \quad \underbrace{\left\langle \Omega \middle| \mathcal{T} A^\mu(y_2)\psi(x_2)A^\nu(y_1)\bar{\psi}(x_1) \middle| \Omega \right\rangle}_{=:\mathcal{G}} \quad \left(i\overleftarrow{\not{\partial}}_{x_1} + m\right) \quad u_{p_1} \quad e^{-ip_1\cdot x_1}.
\end{aligned}
$$

$\mathcal{G}$ is now our Green's function or n-point function (actually, 4-point function).

TURN THE FIELDS IN THE GREEN'S FUNCTION INTO INTERACTION PICTURE FIELDS:

We saw in the section about n-point function, that we can turn $\mathcal{G}$ into

$$\mathcal{G} = \frac{\langle 0 | \mathcal{T} \quad A^\mu(y_2)\psi(x_2)A^\nu(y_1)\bar{\psi}(x_1) \quad \exp\left(i \int d^4z\, \mathcal{L}_{\mathrm{Int},I}\right) |0\rangle}{\langle 0|\mathcal{T} \exp\left(i\int d^4z\, \mathcal{L}_{\mathrm{Int},I}\right)|0\rangle} =: \frac{\mathcal{G}_N}{\mathcal{G}_D}.$$

The fields in the $|\Omega\rangle$-vacuum expectation value where Heisenberg picture fields of the interacting theory. As we discussed in (>7.9.1), the fields of this formula with $|0\rangle$-vacuum expectation values are interaction picture fields of the interacting theory and thereby Heisenberg picture fields of the free theory (>7.9.2). In the formula above we did not mark this in any way (for example by writing $A_{I\mu}, \psi_I$ instead of A_μ, ψ). From now on, starting at the formula above, all the fields will be understood to be in the interaction picture, despite we don't mark them in that way!

PERTURBATION THEORY: EXPANSION OF THE EXPONENTIAL:

Let's consider the numerator $\mathcal{G}_N$ of $\mathcal{G}$. We can expand the exponential like

$$\exp\left(i\int d^4z\, \mathcal{L}_{\mathrm{Int},I}\right) = 1 + i\int d^4z\, \mathcal{L}_{\mathrm{Int},I} - \frac{1}{2!}\int d^4z_1\, d^4dz_2\, \mathcal{L}_{\mathrm{Int},I}(z_1)\, \mathcal{L}_{\mathrm{Int},I}(z_2) + \cdots.$$

In zeroth order we have no interaction. If we demand that $p_1 \neq p_2$, $k_1 \neq k_2$, we physically know that the zeroth order will not contribute, because without interaction the momenta cannot change. The first order will also not contribute because Wick's theorem told us that a vacuum expectation value of an odd number of fields vanishes. $\mathcal{L}_{\mathrm{Int},I}$ contains three fields, together with the four fields outside the exponential we have seven, which is odd.[1] We therefore will only consider the second order.

Since we have neglected the term without any interaction, what we are going to calculate is (>8.2.2)

$$\begin{aligned} \mathcal{S} &:= \langle \{k_2, p_2\}_+ | \{k_1, p_1\}_- \rangle = \langle \{k_2, p_2\}_+ |S| \{k_1, p_1\}_- \rangle \\ &= \underbrace{\langle \{k_2, p_2\}_+ | \{k_1, p_1\}_- \rangle}_{\text{neglected}} + \langle \{k_2, p_2\}_+ |iT| \{k_1, p_1\}_- \rangle \mathrel{\widehat{=}} \langle \{k_2, p_2\}_+ |iT| \{k_1, p_1\}_- \rangle \\ &= (2\pi)\delta(p_1 + k_1 - p_2 - k_2) \cdot i\mathcal{M}. \end{aligned}$$

PHOTON FIELDS:

In second order perturbation theory and after plugging in the interacting Lagrangian, our numerator $\mathcal{G}_N$ takes on the form

$$\begin{aligned} \mathcal{G}_N &= -\frac{1}{2}\int d^4z_1\, d^4z_2\, \langle 0|\mathcal{T} \quad A^\mu(y_2)\psi(x_2)A^\nu(y_1)\bar{\psi}(x_1)\ \ \mathcal{L}_{\mathrm{Int},I}(z_1)\, \mathcal{L}_{\mathrm{Int},I}(z_2)|0\rangle \\ &= -\frac{g^2}{2}\int d^4z_1\, d^4z_2 \\ &\qquad \langle 0|\mathcal{T} \quad A^\mu(y_2)\psi(x_2)A^\nu(y_1)\bar{\psi}(x_1)\ \ \bar{\psi}(z_1)\gamma_\sigma\psi(z_1)A^\sigma(z_1)\ \ \bar{\psi}(z_2)\gamma_\kappa\psi(z_2)A^\kappa(z_2)|0\rangle. \end{aligned}$$

The photon fields commute with the fermion fields and we can separate them, as we saw in (>7.10.2):

$$\begin{aligned} \mathcal{G}_N &= -\frac{g^2}{2}\int d^4z_1\, d^4z_2 \\ &\qquad \langle 0|\mathcal{T}A^\mu(y_2)A^\nu(y_1)A^\sigma(z_1)A^\kappa(z_2)|0\rangle\ \ \langle 0|\mathcal{T}\psi(x_2)\bar{\psi}(x_1)\bar{\psi}(z_1)\gamma_\sigma\psi(z_1)\bar{\psi}(z_2)\gamma_\kappa\psi(z_2)|0\rangle. \end{aligned}$$

Consider first the photon fields only. Wick's Theorem gives us now, using $\widehat{D}^{\mu\nu}_{x_i y_j} := \widehat{D}^{\mu\nu}_F(x_i - y_j)$,

[1] Technically, what is more important is the number of fields of *the same type*. However, if the total number of fields is odd, also the number of at least one field type is odd and therefore vanishes. Since 2-point functions are multiplied according to Wick's theorem, we get no contribution at all.

$$\langle 0|\mathcal{T}A^{\mu}(y_2)A^{\nu}(y_1)A^{\sigma}(z_1)A^{\kappa}(z_2)|0\rangle = \widehat{D}^{\mu\nu}_{y_2y_1}\widehat{D}^{\sigma\kappa}_{z_1z_2} + \widehat{D}^{\mu\sigma}_{y_2z_1}\widehat{D}^{\nu\kappa}_{y_1z_2} + \widehat{D}^{\mu\kappa}_{y_2z_2}\widehat{D}^{\nu\sigma}_{y_1z_1}.$$

We will now see that the first term vanishes.[1] We consider now only the first term, plug it back into $\mathcal{G}_N$, plug $\mathcal{G}_N$ back into $\mathcal{G}$ and finally plug $\mathcal{G}$ back into $\mathcal{M}_1$ (since it is the first term only, we use the index 1). Now we consider the y_i-integrals only, pull everything which is y_i-independent in front of it and treat is as a proportionality factor:

$$\begin{aligned}\mathcal{S}_1 &\sim \int d^4y_2\, d^4y_1 \quad e^{ik_2\cdot y_2}e^{-ik_1\cdot y_1}\Box_{y_2}\Box_{y_1}\widehat{D}^{\mu\nu}_F(y_2-y_1)\\ &= \int d^4y_2\, d^4y_1 \quad e^{ik_2\cdot y_2}e^{-ik_1\cdot y_1}\Box_{y_2}\Box_{y_1}\int d^4\bar{q}\,\frac{-i\eta^{\mu\nu}}{q^2+i\epsilon}e^{-iq\cdot(y_2-y_1)},\end{aligned}$$

where we plugged in the integral form of the Feynman propagator. When we now apply the quablas, we get down a factor of $(-iq)^2(iq)^2 = q^4$ from the exponential. Using that $q^4/(q^2+i\epsilon) \to q^2$ in the limes $\epsilon \to 0$, we are left with (neglecting also all constant factors of $\widehat{D}^{\mu\nu}_F$):

$$\begin{aligned}\mathcal{S}_1 &\sim \int d^4y_2\, d^4y_1\, d^4\bar{q}\; q^2\, e^{ik_2\cdot y_2}e^{-ik_1\cdot y_1}e^{-iq\cdot(y_2-y_1)} = \int d^4y_2\, d^4y_1\, d^4\bar{q}\; q^2\, e^{i(k_2-q)\cdot y_2}e^{-i(k_1-q)\cdot y_1}\\ &= (2\pi)^8\int d^4\bar{q}\; q^2\,\delta(k_2-q)\delta(k_1-q) = (2\pi)^4 k_2^2\,\delta(k_1-k_2) = 0\end{aligned}$$

for $k_2 \neq k_1$.

In the same way we will now consider $\mathcal{S}_{2,3} \coloneqq \mathcal{S}_2 + \mathcal{S}_3$, i.e. the terms $\widehat{D}^{\mu\sigma}_{y_2z_1}\widehat{D}^{\nu\kappa}_{y_1z_2} + \widehat{D}^{\mu\kappa}_{y_2z_2}\widehat{D}^{\nu\sigma}_{y_1z_1}$:[2]

$$\begin{aligned}\mathcal{S}_{2,3} \sim \int d^4y_2\, d^4y_1 \quad & e^{ik_2\cdot y_2}e^{-ik_1\cdot y_1} \quad \varepsilon_{k_2\mu}\varepsilon_{k_1\nu} \quad \Box_{y_2}\Box_{y_1}\\ & \Big(\widehat{D}^{\mu\sigma}_F(y_2-z_1)\widehat{D}^{\nu\kappa}_F(y_1-z_2) + \widehat{D}^{\mu\kappa}_F(y_2-z_2)\widehat{D}^{\nu\sigma}_F(y_1-z_1)\Big).\end{aligned}$$

Note that $\mathcal{G}_N$ and therefore also $\mathcal{S}_{2,3}$ contains also integrals over z_1 and z_2, which are not written out here but absorbed into the proportionality sign. Still, this fact allows us to relabel the integration variables $z_1 \leftrightarrow z_2$ and the summation indices $\kappa \leftrightarrow \sigma$ *for the first term in the brackets*. Obviously, the two terms in the brackets are now equal and we get a factor of 2:[3]

$$\mathcal{S}_{2,3} \sim 2\int d^4y_2\, d^4y_1 \quad e^{ik_2\cdot y_2}e^{-ik_1\cdot y_1} \quad \varepsilon_{k_2\mu}\varepsilon_{k_1\nu} \quad \Box_{y_2}\Box_{y_1} \quad \widehat{D}^{\mu\kappa}_F(y_2-z_2)\widehat{D}^{\nu\sigma}_F(y_1-z_1).$$

We can now plug in the integral formulas of the Feynman propagators and apply the quablas:

$$\begin{aligned}\mathcal{S}_{2,3} \sim 2\int d^4y_2\, d^4y_1 \quad & e^{ik_2\cdot y_2}e^{-ik_1\cdot y_1} \quad \varepsilon_{k_2\mu}\varepsilon_{k_1\nu} \quad \Box_{y_2}\Box_{y_1}\\ & \int d^4\bar{q}_1\,\frac{-i\eta^{\mu\kappa}}{q_1^2+i\epsilon}e^{-iq_1\cdot(y_2-z_2)}\int d^4\bar{q}_2\,\frac{-i\eta^{\nu\sigma}}{q_2^2+i\epsilon}e^{-iq_2\cdot(y_1-z_1)}\end{aligned}$$

---

[1] The z_i-coordinates come from the interaction integral, they can be interpreted at the space-time points at which the interaction take places. All possible interaction space-time points are then integrated over: this is just the superposition principle of quantum mechanics. Anyway, the propagator $\widehat{D}^{\sigma\kappa}_{z_1z_2}$ can be interpreted as a photon line from interaction point z_1 to interaction point z_2 – an internal photon! But for Compton scattering, we do not have any internal photons, because this term vanishes.

[2] Technically, the proportionality sign is not correct at this point, because we have z_i-dependent terms here, but absorbed a z_i-integral into the proportionality sign.

[3] One might worry at this point about the z_i-dependence of the fermion fields. Without interchanging $z_1 \leftrightarrow z_2$ and $\sigma \leftrightarrow \kappa$ (i.e. for the second term in the brackets), we had

$$\big\langle 0\big|\mathcal{T}\psi(x_2)\bar{\psi}(x_1)\bar{\psi}(z_1)\gamma_\sigma\psi(z_1)\bar{\psi}(z_2)\gamma_\kappa\psi(z_2)\big|0\big\rangle,$$

but with interchanging (i.e. for the first term in the brackets) this becomes

$$\big\langle 0\big|\mathcal{T}\psi(x_2)\bar{\psi}(x_1)\bar{\psi}(z_2)\gamma_\kappa\psi(z_2)\bar{\psi}(z_1)\gamma_\sigma\psi(z_1)\big|0\big\rangle.$$

However, as they are time-ordered, those two expressions are the same. That is, we can place this term in front of the brackets and the manipulation we did are valid.

$$
\begin{aligned}
&= 2\int d^4y_2\, d^4y_1 \;\; e^{ik_2\cdot y_2}e^{-ik_1\cdot y_1} \;\; \varepsilon_{k_2\mu}\varepsilon_{k_1\nu} \\
&\qquad\qquad (-i)^2(-i\eta^{\mu\kappa}) \int d^4\bar{q}_1\, e^{-iq_1\cdot(y_2-z_2)}\, (-i)^2(-i\eta^{\nu\sigma}) \int d^4\bar{q}_2\, e^{-iq_2\cdot(y_1-z_1)}. \\
&= -2\eta^{\mu\kappa}\eta^{\nu\sigma} \int d^4y_2\, d^4y_1 \;\; e^{ik_2\cdot y_2}e^{-ik_1\cdot y_1} \;\; \varepsilon_{k_2\mu}\varepsilon_{k_1\nu} \;\; \delta(y_2-z_2)\delta(y_1-z_1) \\
&= -2e^{ik_2\cdot z_2}e^{-ik_1\cdot z_1} \;\; \varepsilon^{\kappa}_{k_2}\varepsilon^{\sigma}_{k_1}.
\end{aligned}
$$

That's all about the photons. With this result, our matrix element reads

$$
\begin{aligned}
\mathcal{S} = g^2 i(-i)^3 &\int d^4x_2\, d^4x_1\, d^4z_1\, d^4z_2 \;\; e^{ik_2\cdot z_2}e^{ip_2\cdot x_2}e^{-ik_1\cdot z_1} \;\; \varepsilon^{\kappa}_{k_2}\varepsilon^{\sigma}_{k_1} \\
&\bar{u}_{p_2}\big(i\partial\!\!\!/_{x_2} - m\big)\frac{\langle 0|\mathcal{T}\psi(x_2)\bar{\psi}(x_1)\bar{\psi}(z_1)\gamma_\sigma\psi(z_1)\bar{\psi}(z_2)\gamma_\kappa\psi(z_2)|0\rangle}{\mathcal{G}_D}\big(i\overleftarrow{\partial\!\!\!/}_{x_1} + m\big)u_{p_1} \;\; e^{-ip_1\cdot x_1}.
\end{aligned}
$$

FERMION FIELDS:

We now apply Wick's Theorem to the fermion fields. Recall (in a very hand-waving notation) that the fields contain ladder operators in the way $\psi \sim \big(b^\dagger + a\big)$ and $\bar{\psi} \sim \big(a^\dagger + b\big)$. We also found that *only* $\{a, a^\dagger\}$ and $\{b, b^\dagger\}$ are non-vanishing. Therefore, 2-point functions of $\psi\psi$ and $\bar{\psi}\bar{\psi}$ will always vanish. In the notation of the proof of Wick's theorem, this is very easy to see for the ψ case,

$$
\langle 0|\psi\psi|0\rangle = \langle 0|\psi\psi^+|0\rangle = \langle 0|\psi^+\psi|0\rangle = 0,
$$

and in the same way also for the $\bar{\psi}$ case. Thus, we can only get 2-point functions of $\psi\bar{\psi}$ (or $\bar{\psi}\psi$):

$$
\begin{aligned}
&\langle 0|\mathcal{T}\psi(x_2)\bar{\psi}(x_1)\bar{\psi}(z_1)\psi(z_1)\bar{\psi}(z_2)\psi(z_2)|0\rangle \\
&\quad = \widetilde{D}_{x_2x_1}\widetilde{D}_{z_1z_1}\widetilde{D}_{z_2z_2} + \widetilde{D}_{x_2x_1}\widetilde{D}_{z_2z_1}\widetilde{D}_{z_1z_2} + \widetilde{D}_{x_2z_1}\widetilde{D}_{z_1x_1}\widetilde{D}_{z_2z_2} + \widetilde{D}_{x_2z_1}\widetilde{D}_{z_2x_1}\widetilde{D}_{z_1z_2} \\
&\quad + \widetilde{D}_{x_2z_2}\widetilde{D}_{z_1x_1}\widetilde{D}_{z_2z_1} + \widetilde{D}_{x_2z_2}\widetilde{D}_{z_2x_1}\widetilde{D}_{z_1z_1},
\end{aligned}
$$

where, again, $\widetilde{D}_{x_ix_j} \coloneqq \widetilde{D}_F\big(x_i - x_j\big)$. The first index comes from the ψ field, the second from the $\bar{\psi}$ field. We neglected the γ-matrices here, but we say a few words about them soon.

Moreover, terms including a factor $\widetilde{D}_{x_2x_1}$, i.e. where both variables are x_i-variables, vanish as well, when we apply the Dirac operators in $\mathcal{M}$ (this is analogous to the photon field):

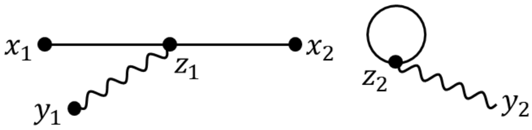
$$
\begin{aligned}
&\int d^4x_2\, d^4x_1 \;\; e^{ip_2\cdot x_2} \;\; \big(i\partial\!\!\!/_{x_2} - m\big)\widetilde{D}_F(x_2 - x_1)\big(i\overleftarrow{\partial\!\!\!/}_{x_1} + m\big) \;\; e^{-ip_1\cdot x_1} \\
&\quad = \int d^4x_2\, d^4x_1 \;\; e^{ip_2\cdot x_2} \;\; \big(i\partial\!\!\!/_{x_2} - m\big)\int d^4\bar{q}\,\frac{i}{q\!\!\!/ - m + i\epsilon}e^{-iq\cdot(x_2-x_1)}\,\big(i\overleftarrow{\partial\!\!\!/}_{x_1} + m\big) \;\; e^{-ip_1\cdot x_1} \\
&\quad = \int d^4x_2\, d^4x_1 \;\; e^{ip_2\cdot x_2} \;\; \int d^4\bar{q}\,\frac{(q\!\!\!/ - m)i(-q\!\!\!/ + m)}{q\!\!\!/ - m + i\epsilon}e^{-iq\cdot(x_2-x_1)} \;\; e^{-ip_1\cdot x_1} \\
&\quad = -i\int d^4x_2\, d^4x_1\, d^4\bar{q}\;(q\!\!\!/ - m)\; e^{i(p_2-q)\cdot x_2}e^{-i(p_1-q)\cdot x_1} \\
&\quad = -i(2\pi)^8\int d^4\bar{q}\;(q\!\!\!/ - m)\;\delta(p_2 - q)\delta(p_1 - q) = -i(2\pi)^4(p\!\!\!/_2 - m)\;\delta(p_1 - p_2) = 0
\end{aligned}
$$

for $p_1 \neq p_2$.

We also will not take the terms with $\widetilde{D}_{z_1z_1}$ and $\widetilde{D}_{z_2z_2}$ into account, by a physical argument, similar to the one we gave for neglecting the zeroth order in the expansion of the exponential. For *photons*, we had two non-vanishing terms after applying Wick's theorem, which turned out to be the same and we were left with the photon propagators $\widehat{D}^{\mu\kappa}_{y_zz_2}\widehat{D}^{\nu\sigma}_{y_1z_1}$ only. All the propagators appearing, for example, in the $\widetilde{D}_{z_2z_2}$-term are

$$
\widetilde{D}_{x_2z_1}\widetilde{D}_{z_1x_1}\widetilde{D}_{z_2z_2}\widehat{D}^{\mu\kappa}_{y_zz_2}\widehat{D}^{\nu\sigma}_{y_1z_1}.
$$

The indices are interpreted as vertices (space-time points, where interactions take place) and the propagators connect there. Thus, those four propagators, yield the following Feynman diagram:



This diagram is disconnected! Indeed, the right-hand part contains the propagator $\widetilde{D}_{z_2z_2}$, which is divergent. This process can happen anytime, anywhere. We are not interested in such processes and we will leave them aside. More rigorously, they are cancelled by the denominator $\mathcal{G}_D$, as we saw quite explicitly for the ϕ^4-theory. We will not proof this for the Compton scattering case, but rather accept that the disconnected diagrams cancel $\mathcal{G}_D$.

All that is left is

$$\mathcal{S} = g^2 i(-i)^3 \int d^4x_2\, d^4x_1\, d^4z_1\, d^4z_2 \quad e^{ik_2\cdot z_2} e^{ip_2\cdot x_2} e^{-ik_1\cdot z_1} \quad \varepsilon^{\kappa}_{k_2} \varepsilon^{\sigma}_{k_1}$$
$$\bar{u}_{p_2}\big(i\not{\partial}_{x_2} - m\big)\big(\widetilde{D}_{x_2z_1}\gamma_\sigma \widetilde{D}_{z_1z_2}\gamma_\kappa \widetilde{D}_{z_2x_1} + \widetilde{D}_{x_2z_2}\gamma_\kappa \widetilde{D}_{z_2z_1}\gamma_\sigma \widetilde{D}_{z_1x_1}\big)\big(i\overleftarrow{\not{\partial}}_{x_1} + m\big)u_{p_1} \quad e^{-ip_1\cdot x_1}.$$

Note that the fermionic Feynman propagators are matrices and they do not commute with each other nor with the γ-matrices. That the order has to be that way becomes obvious when writing down indices:

$$\begin{aligned}
&\left\langle 0\middle|\mathcal{T}\psi^a(x_2)\bar{\psi}^b(x_1)\bar{\psi}^c(z_1)\gamma^{cd}_\sigma \psi^d(z_1)\bar{\psi}^e(z_2)\gamma^{ef}_\kappa \psi^f(z_2)\middle|0\right\rangle \\
&\quad = \gamma^{cd}_\sigma \gamma^{ef}_\kappa \big(\widetilde{D}^{ac}_{x_2z_1}\widetilde{D}^{fb}_{z_2x_1}\widetilde{D}^{de}_{z_1z_2} + \widetilde{D}^{ae}_{x_2z_2}\widetilde{D}^{db}_{z_1x_1}\widetilde{D}^{fc}_{z_2z_1}\big) \\
&\quad = \widetilde{D}^{ac}_{x_2z_1}\gamma^{cd}_\sigma \widetilde{D}^{de}_{z_1z_2}\gamma^{ef}_\kappa \widetilde{D}^{fb}_{z_2x_1} + \widetilde{D}^{ae}_{x_2z_2}\gamma^{ef}_\kappa \widetilde{D}^{fc}_{z_2z_1}\gamma^{cd}_\sigma \widetilde{D}^{db}_{z_1x_1}.
\end{aligned}$$

Now we apply the derivatives on the propagators,

$$\begin{aligned}
\big(i\not{\partial}_{x_2} - m\big)\widetilde{D}_{x_2z_i} &= \big(i\not{\partial}_{x_2} - m\big)\int d^4\bar{p}\,\frac{i}{\not{p} - m + i\epsilon}e^{-ip\cdot(x_2-z_i)} = \int d^4\bar{p}\,\frac{i(\not{p} - m)}{\not{p} - m + i\epsilon}e^{-ip\cdot(x_2-z_i)} \\
&= i\delta(x_2 - z_i), \\
\widetilde{D}_{z_ix_1}\big(i\overleftarrow{\not{\partial}}_{x_1} + m\big) &= \int d^4\bar{p}\,\frac{i}{\not{p} - m + i\epsilon}e^{-ip\cdot(z_i-x_1)}\big(i\overleftarrow{\not{\partial}}_{x_1} + m\big) = \int d^4\bar{p}\,\frac{i(-\not{p} + m)}{\not{p} - m + i\epsilon}e^{-ip\cdot(z_i-x_1)} \\
&= -i\delta(z_i - x_1),
\end{aligned}$$

and get

$$\begin{aligned}
\mathcal{S} &= g^2 i(-i)^3 \int d^4x_2\, d^4x_1\, d^4z_1\, d^4z_2 \quad e^{ik_2\cdot z_2}e^{ip_2\cdot x_2}e^{-ik_1\cdot z_1} \quad \varepsilon^{\kappa}_{k_2}\varepsilon^{\sigma}_{k_1} \\
&\qquad \bar{u}_{p_2}\Big(-i\delta(x_2 - z_1)\gamma_\sigma \widetilde{D}_{z_1z_2}\gamma_\kappa i\delta(z_2 - x_1) - i\delta(x_2 - z_2)\gamma_\kappa \widetilde{D}_{z_2z_1}\gamma_\sigma i\delta(z_1 - x_1)\Big)u_{p_1} \quad e^{-ip_1\cdot x_1} \\
&= -g^2 \int d^4z_1\, d^4z_2 \quad e^{ik_2\cdot z_2}e^{-ik_1\cdot z_1} \\
&\qquad \bar{u}_{p_2}\big(e^{ip_2\cdot z_1}e^{-ip_1\cdot z_2}\not{\varepsilon}_{k_1}\widetilde{D}_{z_1z_2}\not{\varepsilon}_{k_2} + e^{ip_2\cdot z_2}e^{-ip_1\cdot z_1}\not{\varepsilon}_{k_2}\widetilde{D}_{z_2z_1}\not{\varepsilon}_{k_1}\big)u_{p_1} \\
&= -g^2 \int d^4z_1\, d^4z_2\, d^4\bar{q} \quad e^{ik_2\cdot z_2}e^{-ik_1\cdot z_1} \\
&\qquad \bar{u}_{p_2}\left(e^{ip_2\cdot z_1}e^{-ip_1\cdot z_2}\not{\varepsilon}_{k_1}\frac{ie^{-iq\cdot(z_1-z_2)}}{\not{q} - m + i\epsilon}\not{\varepsilon}_{k_2} + e^{ip_2\cdot z_2}e^{-ip_1\cdot z_1}\not{\varepsilon}_{k_2}\frac{ie^{-iq\cdot(z_2-z_1)}}{\not{q} - m + i\epsilon}\not{\varepsilon}_{k_1}\right)u_{p_1} \\
&= -g^2(2\pi)^4 \int d^4q \\
&\qquad \bar{u}_{p_2}\left(\not{\varepsilon}_{k_1}\frac{i\delta(p_2 - k_1 - q)\delta(k_2 - p_1 + q)}{\not{q} - m + i\epsilon}\not{\varepsilon}_{k_2} + \not{\varepsilon}_{k_2}\frac{i\delta(q - k_1 - p_1)\delta(k_2 + p_2 - q)}{\not{q} - m + i\epsilon}\not{\varepsilon}_{k_1}\right)u_{p_1} \\
&= -g^2(2\pi)^4\delta(k_2 + p_2 - k_1 - p_1)
\end{aligned}$$

$$\bar{u}_{p_2}\left(\varepsilon_{k_1}\frac{i}{(\not{p}_2-\not{k}_1)-m+i\epsilon}\varepsilon_{k_2}+\varepsilon_{k_2}\frac{i}{(\not{k}_1+\not{p}_1)-m+i\epsilon}\varepsilon_{k_1}\right)u_{p_1}.$$

As explained earlier in this section, the relation between $\mathcal{S}$ and $i\mathcal{M}$ reads

$$\mathcal{S} \mathrel{\widehat{=}} (2\pi)\delta(p_1+k_1-p_2-k_2)\cdot i\mathcal{M},$$

as long as we leave non-interacting diagrams aside (what we always do). If we compare this to the equation above, we can extract $i\mathcal{M}$ to be exactly the same as in (>8.3.1):

$$i\mathcal{M} = -g^2\bar{u}_{p_2}\left(\varepsilon_{k_2}\frac{i}{(\not{p}_1+\not{k}_1)-m+i\epsilon}\varepsilon_{k_1}+\varepsilon_{k_1}\frac{i}{(\not{p}_1-\not{k}_2)-m+i\epsilon}\varepsilon_{k_2}\right)u_{p_1}.$$

9 CROSS SECTIONS AND DECAY RATES

9.1 Scattering Probability

9.1.1 Position Space Wave Function yields Probability Interpretation

The position space wave function[1]

$$\tilde{f}(x) \coloneqq \int d\tilde{p}\, e^{-ip\cdot x} f(p),$$

which satisfies the Klein-Gordon equation[2]

$$(\Box + m^2)\tilde{f}(x) = \int d\tilde{p}\, (-p^2 + m^2) e^{-ip\cdot x} f(p) = 0$$

can be used to define the probability current

$$j^\mu(x) = i\tilde{f}^*(x)\overleftrightarrow{\partial}^\mu f(x), \quad \overleftrightarrow{\partial}^\mu \coloneqq \partial_\mu - \overleftarrow{\partial}_\mu.$$

Consider[3]

$$\begin{aligned}
\int d^3x\, j^\mu(x) &= i\int d^3x\, \tilde{f}^*(x)\overleftrightarrow{\partial}^\mu f(x) = i\int d^3x\, d\tilde{p}_1\, d\tilde{p}_2 \;\; f^*(p_1) f(p_2) \;\; e^{ip_1\cdot x}\overleftrightarrow{\partial}^\mu e^{-ip_2\cdot x} \\
&= i\int d^3x\, d\tilde{p}_1\, d\tilde{p}_2 \;\; f^*(p_1) f(p_2) \;\; \left(-ip_2^\mu - ip_1^\mu\right) \;\; e^{ip_1\cdot x} e^{-ip_2\cdot x} \\
&= \int d\tilde{p}_1\, d\tilde{p}_2 \;\; f^*(p_1) f(p_2) \;\; \left(p_1^\mu + p_2^\mu\right) \;\; (2\pi)^3 \delta(\vec{p}_1 - \vec{p}_2) e^{i(\omega_{p_1} - \omega_{p_2})\cdot t} \\
&= \int d\tilde{p}_1 \frac{1}{2\omega_{p_1}} \;\; |f(p_1)|^2 \;\; \left(2p_1^\mu\right) = \int d\tilde{p}\, \frac{p^\mu}{\omega_p} |f(p)|^2.
\end{aligned}$$

The zeroth component of the current can now be interpreted as a probability density $\rho(x) \coloneqq j^0(x)$, since

$$\int d^3x\, \rho(x) = \int d\tilde{p}\;\, \underbrace{p^0/\omega_p}_{=1} \;\, |f(p)|^2 = 1,$$

if the wave functions in momentum space $f(p)$ are properly normalized.

9.1.2 Formula for the Transition Probability

For simplicity, let's see how this works for only two incoming momenta,

$$|i\rangle = \int d\tilde{p}_1\, d\tilde{p}_2\, f_1(p_1) f_2(p_2) |p_1, p_2\rangle.$$

This will be straight forward to generalize. Note that we leave the "out" state $|f\rangle$ completely arbitrary. The probability for starting with state $|i\rangle$ and ending up with the state $|f\rangle$ reads

---

[1] This looks like a Fourier transformation, but actually that is not quite the case, since we have the differential

$$d\tilde{p} = \frac{d^3p}{(2\pi)^3 2\omega_p} \quad \text{instead of} \quad d^4\bar{p} = \frac{d^4p}{(2\pi)^4}.$$

But this does not prevent us from defining $\tilde{f}(x)$ in such a way.

[2] Recall $p^2 = E^2 - \vec{p}^2 = m^2$.

[3] Note, that $p^\mu = (\omega_p, \vec{p})$ and $\omega_p = \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. Thus, it is justified to turn the 4-vector p_2^μ into p_1^μ with the 3-vector δ-function $\delta(\vec{p}_1 - \vec{p}_2)$.

$$
\begin{aligned}
w_{fi} &= \left|S_{fi}\right|^2 = \langle f|S|i\rangle\langle f|S|i\rangle^* \\
&= \int d\tilde{p}_1\, d\tilde{p}_2\, d\tilde{p}_1'\, d\tilde{p}_2' \quad f_1(p_1) f_2(p_2) f_1^*(p_1') f_2^*(p_2') \;\; \langle f|S|p_1, p_2\rangle \langle f|S|p_1', p_2'\rangle^*.
\end{aligned}
$$

We saw in (>8.3.1) that any S-matrix element contains a δ-function of total 4-momentum conservation. Let's make this explicit by writing,

$$
\langle f|S|p_1, p_2\rangle = (2\pi)^4 \delta\left(p_f - p_1 - p_2\right) i\mathcal{M}\left(\{p_f\}, p_1, p_2\right),
$$

where p_f is the sum of all outgoing momenta and $\{p_f\}$ is the set of them. $i\mathcal{M}\left(\{p_f\}, p_1, p_2\right)$ is the *amplitude*, that is simply the expression we get from Feynman rules *without* the factor of $(2\pi)^4\delta\left(p_f - p_1 - p_2\right)$ (>8.2.2) Using this notation, we find

$$
\begin{aligned}
w_{fi} = \int d\tilde{p}_1\, d\tilde{p}_2\, d\tilde{p}_1'\, d\tilde{p}_2' \quad & f_1(p_1) f_2(p_2) f_1^*(p_1') f_2^*(p_2') \\
& (2\pi)^8 \delta\left(p_f - p_1 - p_2\right) \delta\left(p_f - p_1' - p_2'\right) \;\; \mathcal{M}\left(\{p_f\}, p_1, p_2\right) \mathcal{M}^*\left(\{p_f\}, p_1', p_2'\right).
\end{aligned}
$$

The expression in the integral is non-zero only for $p_f = p_1 + p_2$, due to the first δ-function. Thus, we can write $p_f = p_1 + p_2$ in the second δ-function and then we can write it as an integral over an exponential function (its Fourier transform):

$$
\begin{aligned}
w_{fi} &= \int d\tilde{p}_1\, d\tilde{p}_2\, d\tilde{p}_1'\, d\tilde{p}_2' \quad f_1(p_1) f_2(p_2) f_1^*(p_1') f_2^*(p_2') \\
&\qquad (2\pi)^8 \delta\left(p_f - p_1 - p_2\right) \delta\left(p_1 + p_2 - p_1' - p_2'\right) \;\; \mathcal{M}\left(\{p_f\}, p_1, p_2\right) \mathcal{M}^*\left(\{p_f\}, p_1', p_2'\right) \\
&= \int d\tilde{p}_1\, d\tilde{p}_2\, d\tilde{p}_1'\, d\tilde{p}_2'\, d^4x \quad f_1(p_1) f_2(p_2) f_1^*(p_1') f_2^*(p_2') \;\; e^{-i\left(p_1 + p_2 - p_1' - p_2'\right)\cdot x} \\
&\qquad (2\pi)^4 \delta\left(p_f - p_1 - p_2\right) \;\; \mathcal{M}\left(\{p_f\}, p_1, p_2\right) \mathcal{M}^*\left(\{p_f\}, p_1', p_2'\right) \\
&= \int d\tilde{p}_1\, d\tilde{p}_2\, d\tilde{p}_1'\, d\tilde{p}_2'\, d^4x \quad f_1(p_1) f_2(p_2) f_1^*(p_1') f_2^*(p_2') \;\; e^{-i\left(p_1 + p_2 - p_1' - p_2'\right)\cdot x} \\
&\qquad (2\pi)^4 \delta\left(p_f - \bar{p}_1 - \bar{p}_2\right) \;\; \mathcal{M}\left(\{p_f\}, \bar{p}_1, \bar{p}_2\right) \mathcal{M}^*\left(\{p_f\}, \bar{p}_1, \bar{p}_2\right)
\end{aligned}
$$

In the last step, we assumed that the momenta do not vary too much over the width of the wave packages $f_i(p_i)$. In this case, we can take the momenta in the amplitude $\mathcal{M}$ and the remaining δ-function as the constant average momenta $\bar{p}_i$. Note that the wave packages of the dashed momenta are the same as for the undashed, such that they have the same average value. We did not do so in the exponential function, because we assumed that the exponential varies stronger with the momenta than the amplitude.

Let's now define "some kind of" a Fourier transform[1]

$$
\tilde{f}_i(x) := \int d\tilde{p}\; e^{-ip\cdot x} f_i(p).
$$

As we saw in (>9.1.1), this can be interpreted as a position space wave function. Since those $d\tilde{p}$-integrals appear four times in the expression for w_{fi} we derived so far, we get four times a factor of $\tilde{f}_i(x)$:

$$
\begin{aligned}
& w_{fi} = \int d^4x \;\; \left|\tilde{f}_1(x)\right|^2 \left|\tilde{f}_2(x)\right|^2 \;\; (2\pi)^4 \delta\left(p_f - \bar{p}_1 - \bar{p}_2\right) \;\; \left|\mathcal{M}\left(\{p_f\}, \bar{p}_1, \bar{p}_2\right)\right|^2 \\
\Leftrightarrow \quad & \frac{d^4 w_{fi}}{d^4 x} = \frac{d^4 w_{fi}}{d^3x\, dt} = \left|\tilde{f}_1(x)\right|^2 \left|\tilde{f}_2(x)\right|^2 \;\; (2\pi)^4 \delta\left(p_f - \bar{p}_1 - \bar{p}_2\right) \;\; \left|\mathcal{M}\left(\{p_f\}, \bar{p}_1, \bar{p}_2\right)\right|^2.
\end{aligned}
$$

For an arbitrary number of incoming momenta, we have obviously

[1] See footnote 16 on page 51.

$$\frac{d^4 w_{fi}}{d^3x\,dt} = (2\pi)^4 \delta(p_f - \bar{p}_i)\ \ \left|\mathcal{M}(\{p_f\},\{\bar{p}_i\})\right|^2\ \ \prod_i \left|\tilde{f}_i(x)\right|^2.$$

9.2 The Cross Section

9.2.1 Derivation of the Differential Cross Section

We start at the definition

$$d\sigma = \frac{\#\text{probability of scattering/time/volume}}{\text{incident flux density} \cdot \#\text{scatterers/volume}}.$$

Consider a scattering of two incoming particles with momenta p_1, p_2. Actually, those momenta are the average momenta over a wave package $\bar{p}_1, \bar{p}_2$ as we introduced them in (>9.1.2). We will just omit writing the bars here. The probability of a scattering event per time and volume is the expression we found in (>9.1.2):

$$\begin{aligned}\#\text{probability of scattering/time/volume} &= \int \{d\tilde{p}_f\} \frac{d^4 w_{fi}}{d^3x\,dt} \\ &= \int \{d\tilde{p}_f\}\ (2\pi)^4 \delta(p_f - p_1 - p_2)\ \ \left|\mathcal{M}(\{p_f\}, p_1, p_2)\right|^2\ \ \left|\tilde{f}_1(x)\right|^2 \left|\tilde{f}_2(x)\right|^2;\end{aligned}$$

w_{fi} was defined as the probability for scattering from a state $|i\rangle$ to a state $|f\rangle$ with momenta $\{p_f\}$. However, here we are interested in the probability for the scattering taking place at all, regardless of the final momenta. Thus, we have to sum up all the possible final momenta by an integration with $\{d\tilde{p}_f\} \coloneqq \Pi_n d\tilde{p}_n$, where the product goes over all momenta of external particles of the final state.

Let's consider the scattering in the rest frame of the second particle, such that it is at rest ($\vec{p}_2 = 0$). In this case, the incident flux contains only the first particle. As we saw in (>9.1.1), the 4-current density can be given in terms of the wave function as

$$j^\mu(x) = i\tilde{f}^*(x)\overleftrightarrow{\partial}^\mu f(x), \quad \tilde{f}(x) \coloneqq \int d\tilde{p}\ e^{-ip\cdot x} f(p).$$

Note, that we can give this current as

$$\begin{aligned}|j^\mu(x)| &= \left|i\tilde{f}^*(x)\overleftrightarrow{\partial}^\mu f(x)\right| = \left|\iint d\tilde{p}\ d\tilde{p}'\ f^*(p)f(p')\ e^{ip\cdot x}\overleftrightarrow{\partial}^\mu e^{-ip'\cdot x}\right| \\ &= \left|\iint d\tilde{p}\ d\tilde{p}'\ f^*(p)f(p')\ (-ip^\mu - ip'^\mu)e^{i(p-p')\cdot x}\right| \\ &= \left|\iint d\tilde{p}\ d\tilde{p}'\ f^*(p)f(p')\ (p^\mu + p'^\mu)e^{i(p-p')\cdot x}\right| \\ &\approx \left|\iint d\tilde{p}\ d\tilde{p}'\ f^*(p)f(p')\ (\bar{p}^\mu + \bar{p}'^\mu)e^{i(p-p')\cdot x}\right| = 2|\bar{p}^\mu| \left|\iint d\tilde{p}\ d\tilde{p}'\ f^*(p)f(p')\ e^{i(p-p')\cdot x}\right| \\ &= 2|\bar{p}^\mu| \left|\tilde{f}(x)\right|^2 = 2|p^\mu| \left|\tilde{f}(x)\right|^2.\end{aligned}$$

In the same way as we did it in (>9.1.2), $\bar{p}^\mu$ is the average momentum of the wave package and when writing $p^\mu + p'^\mu \to \bar{p}^\mu + \bar{p}'^\mu$ we assumed, again, that the momentum does not vary too much over the width of the wave package. Since both p^μ and p'^μ are related to the wave function of the same particle, their average is the same: $2p^\mu = p^\mu + p'^\mu$. In the last step, we omitted writing the bar, but it's still the average momentum of the wave package.

Thus, the incident flux from the first particles can be given as

$$\text{incident flux density} = |\vec{j}_1(x)| = 2|\vec{p}_1|^2 \left|\tilde{f}_1(x)\right|^2$$

and the scatterer density (second particle) as

$$\#\text{scatterers/volume} = |\rho_2(x)| = |j_2^0(x)| = 2|p_2^0||\tilde{f}_2(x)|^2 = 2m_2|\tilde{f}_2(x)|^2,$$

where we used that $p_2^0 = m_2$, since we assumed the second particle to be at rest. Plugging all together, we find

$$\begin{aligned}\sigma &= \int \{d\tilde{p}_f\} \frac{(2\pi)^4 \delta(p_f - p_1 - p_2)|\mathcal{M}(\{p_f\}, p_1, p_2)|^2 |\tilde{f}_1(x)|^2 |\tilde{f}_2(x)|^2}{2|\vec{p}_1|^2 |\tilde{f}_1(x)|^2 \cdot 2m_2 |\tilde{f}_2(x)|^2} \\ &= \frac{(2\pi)^4 \delta(p_f - p_1 - p_2)}{2|\vec{p}_1|^2 \cdot 2m_2} |\mathcal{M}(\{p_f\}, p_1, p_2)|^2.\end{aligned}$$

So far, we have worked in the rest frame of the second particle. In general, we should use the factor

$$\frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \quad \text{instead of} \quad \frac{1}{2|\vec{p}_1|^2 \cdot 2m_2}.$$

This general factor will give the right result in the case of particle two being at rest ($p_1^0 = \omega_{p_1}, p_2^0 = m_2$):

$$\frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} = \frac{1}{4\sqrt{\omega_{p_1}^2 m_2^2 - m_1^2 m_2^2}} = \frac{1}{4m_2^2\sqrt{\vec{p}_1^2 + m_1 - m_1^2}} = \frac{1}{4m_2^2|\vec{p}_1|^2}.$$

Thus, we get our final result for the total cross section

$$\begin{aligned}\sigma &= \int \{d\tilde{p}_f\} \frac{(2\pi)^4 \delta(p_f - p_1 - p_2)|\mathcal{M}(\{p_f\}, p_1, p_2)|^2 |\tilde{f}_1(x)|^2 |\tilde{f}_2(x)|^2}{2|\vec{p}_1|^2 |\tilde{f}_1(x)|^2 \cdot 2m_2 |\tilde{f}_2(x)|^2} \\ &= \int \{d\tilde{p}_f\} \frac{(2\pi)^4 \delta(p_f - p_1 - p_2)}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} |\mathcal{M}(\{p_f\}, p_1, p_2)|^2.\end{aligned}$$

The differential cross section is obviously given by

$$\begin{aligned}d\sigma &= \frac{(2\pi)^4 \delta(p_f - p_1 - p_2)}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} |\mathcal{M}(\{p_f\}, p_1, p_2)|^2 \{d\tilde{p}_f\} \\ &= \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} |\mathcal{M}(\{p_f\}, p_1, p_2)|^2 \, d\phi_n,\end{aligned}$$

which defines the so-called *Lorentz invariant phase space measure* (LIPS) to be

$$d\phi_n \coloneqq \{d\tilde{p}_f\} \, (2\pi)^4 \delta(p_f - p_1 - p_2).$$

9.3 The Decay Rate

9.3.1 Derivation of the Decay Rate

We saw that

$$\Gamma = -\frac{\text{decay probability}}{\text{probability density}},$$

where we know from (>9.1.2) that

$$\text{decay probability} = \int \{d\tilde{p}_f\} \frac{d^4 w_{fi}}{d^3x \, dt} = \int \{d\tilde{p}_f\} \, (2\pi)^4 \delta(p_f - p_i) \ \ |\mathcal{M}(\{p_f\}, p_i)|^2 \ \ |\tilde{f}_i(x)|^2,$$

where p_i is the average momentum of the wave package of the decaying particle (the one we denoted as $\bar{p}_i$ earlier) and $\tilde{f}_i$ is the wave function of the decaying particle. p_f is the sum (and $\{p_f\}$ the set) of all the final particles the decaying particle decayed into. We need to integrate over all final state momenta $\{d\tilde{p}_f\}$, as we are interested into the total decay probability, regardless of the final momenta. As derived in (>9.2.1), the probability density is given as

$$\text{probability density} = |\rho(x)| = |j^0(x)| = 2p_i^0\left|\tilde{f}_i(x)\right|^2.$$

Thus, we find

$$\begin{aligned}\Gamma &= -\int \{d\tilde{p}_f\}\frac{(2\pi)^4\delta\left(p_f - p_i\right)\ \left|\mathcal{M}\left(\{p_f\}, p_i\right)\right|^2\ \left|\tilde{f}_i(x)\right|^2}{2p_i^0\left|\tilde{f}_i(x)\right|^2} \\ &= -\int \{d\tilde{p}_f\}\frac{(2\pi)^4\delta\left(p_f - p_i\right)}{2p_i^0}\left|\mathcal{M}\left(\{p_f\}, p_i\right)\right|^2 = -\int d\phi_n \frac{1}{2p_i^0}\left|\mathcal{M}\left(\{p_f\}, p_i\right)\right|^2,\end{aligned}$$

where in this case the Lorentz invariant phase space measure (LIPS) is defined as

$$d\phi_n = \{d\tilde{p}_f\}\,(2\pi)^4\delta\left(p_f - p_i\right).$$

10.3 Sum over Spins and Polarizations

10.3.1 Sum over Electron Spins

Let's define Γ to be everything in $i\mathcal{M}$, which does not depend on the spins α_i (except for the factor i) such that $i\mathcal{M} = \bar{u}_{\alpha_2 p_2} i\Gamma u_{\alpha_1 p_1}$. Thus, Γ is a 4×4-matrix. We find

$$\sum_{\alpha_1,\alpha_2} |\mathcal{M}|^2 = \sum_{\alpha_1,\alpha_2} \bar{u}_{\alpha_2 p_2}\Gamma u_{\alpha_1 p_1}\left(\bar{u}_{\alpha_2 p_2}\Gamma u_{\alpha_1 p_1}\right)^*.$$

Since $\bar{u}\Gamma u$ is a scalar, we can write $* \to \dagger$. Using $1 = \gamma^0\gamma^0$ and $\gamma^{0\dagger} = \gamma^0$ we get

$$\sum_{\alpha_1,\alpha_2} |\mathcal{M}|^2 = \sum_{\alpha_1,\alpha_2} \bar{u}_{\alpha_2 p_2}\Gamma u_{\alpha_1 p_1}\left(\bar{u}_{\alpha_2 p_2}\Gamma u_{\alpha_1 p_1}\right)^\dagger = \sum_{\alpha_1,\alpha_2} \bar{u}_{\alpha_2 p_2}\Gamma u_{\alpha_1 p_1} u^\dagger_{\alpha_1 p_1}\Gamma^\dagger \bar{u}^\dagger_{\alpha_2 p_2}$$
$$= \sum_{\alpha_1,\alpha_2} \bar{u}_{\alpha_2 p_2}\Gamma u_{\alpha_1 p_1}\bar{u}_{\alpha_1 p_1}\gamma^0\Gamma^\dagger\gamma^0 u_{\alpha_2 p_2}.$$

We now use indices to denote the vector- and matrix-products. For them, we use the Einstein convention. We then can use the completeness of the spinors, $\sum_{\alpha_i}\left(u_{\alpha_i p_i}\right)_j\left(\bar{u}_{\alpha_i p_i}\right)_k = (\not{p}_i + m)_{jk}$:

$$\sum_{\alpha_1,\alpha_2} |\mathcal{M}|^2 = \sum_{\alpha_1,\alpha_2} \left(\bar{u}_{\alpha_2 p_2}\right)_i \Gamma_{ij}\left(u_{\alpha_1 p_1}\right)_j\left(\bar{u}_{\alpha_1 p_1}\right)_k\left(\gamma^0\Gamma^\dagger\gamma^0\right)_{kl}\left(u_{\alpha_2 p_2}\right)_l$$
$$= \underbrace{\sum_{\alpha_2}\left(u_{\alpha_2 p_2}\right)_l\left(\bar{u}_{\alpha_2 p_2}\right)_i}_{=(\not{p}_2+m)_{li}}\Gamma_{ij}\underbrace{\sum_{\alpha_1}\left(u_{\alpha_1 p_1}\right)_j\left(\bar{u}_{\alpha_1 p_1}\right)_k}_{=(\not{p}_1+m)_{jk}}\left(\gamma^0\Gamma^\dagger\gamma^0\right)_{kl}$$
$$= (\not{p}_2 + m)_{li}\Gamma_{ij}(\not{p}_1 + m)_{jk}\left(\gamma^0\Gamma^\dagger\gamma^0\right)_{kl} = \mathrm{Tr}\,(\not{p}_2 + m)\Gamma(\not{p}_1 + m)\gamma^0\Gamma^\dagger\gamma^0.$$

Now, we want to take a closer look at $\gamma^0\Gamma^\dagger\gamma^0$. Γ contains only sums of products of 4-vectors contracted with γ-matrices.[1] Using the general property $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ yields $\gamma^0\not{a}^\dagger\gamma^0 = \not{a}$ for a real 4-vector a and thus

$$\gamma^0(\not{a}\,\not{b}\cdots\not{c})^\dagger\gamma^0 = \gamma^0\not{c}^\dagger\cdots\not{b}^\dagger\not{a}^\dagger\gamma^0 = \gamma^0\not{c}^\dagger\gamma^0\cdots\gamma^0\not{b}^\dagger\gamma^0\gamma^0\not{a}^\dagger\gamma^0 = \not{c}\cdots\not{b}\,\not{a}.$$

Plugging in our Γ, we find

$$\sum_{\alpha_1,\alpha_2} |\mathcal{M}|^2 = g^4\,\mathrm{Tr}\,(\not{p}_2 + m)\left(\not{\varepsilon}_{\lambda_2 k_2}\frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon}\not{\varepsilon}_{\lambda_1 k_1} + \not{\varepsilon}_{\lambda_1 k_1}\frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon}\not{\varepsilon}_{\lambda_2 k_2}\right)$$
$$(\not{p}_1 + m)\left(\not{\varepsilon}_{\lambda_1 k_1}\frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon}\not{\varepsilon}_{\lambda_2 k_2} + \not{\varepsilon}_{\lambda_2 k_2}\frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon}\not{\varepsilon}_{\lambda_1 k_1}\right).$$

10.3.2 Sum over Photon Polarizations

We saw in (>6.3.1) that the sum over the (physical) polarizations of polarizations vectors yields

$$\sum_{\lambda=1,2}\varepsilon^\mu_{\lambda k}\varepsilon^\nu_{\lambda k} = -\eta^{\mu\nu} - \frac{k^\mu k^\nu}{(k\cdot n)^2} + \frac{n^\mu k^\nu + k^\mu n^\nu}{k\cdot n}.$$

[1] Recall at this point that

$$\frac{1}{\not{p} - m + i\epsilon} = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon},$$

as we saw in (>5.5.3).

We already stated back then, that the last two terms of this result will not contribute; they never do in QED. We will see now why. In the sum, the polarization vectors carry a μ and a ν index respectively. In the last two terms, at least one of them is carried by a momentum 4-vector k. So, the last wo terms of this formula somehow turn a ε^μ into a k^μ or a ε^ν into a k^ν. Therefore, it is worth to take a look at the amplitude $\mathcal{M}(\varepsilon_{\lambda_1 k_1} \to k_1)$, where $\varepsilon_{\lambda_1 k_1}$ is replaced by k_1:

$$i\mathcal{M}(\varepsilon_{\lambda_1 k_1} \to k_1) = -ig^2 \bar{u}_{\alpha_2 p_2} \left(\not{\varepsilon}_{\lambda_2 k_2} \frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \not{k}_1 + \not{k}_1 \frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \not{\varepsilon}_{\lambda_2 k_2} \right) u_{\alpha_1 p_1}$$

Let's add two zeros of the form $\not{p}_i - m - (\not{p}_i - m)$:

$$\begin{aligned} i\mathcal{M}(\varepsilon_{\lambda_1 k_1} \to k_1) = -ig^2 \bar{u}_{\alpha_2 p_2} \Bigg(& \not{\varepsilon}_{\lambda_2 k_2} \frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \left(\not{p}_1 + \not{k}_1 - m - (\not{p}_1 - m)\right) \\ & - \left(\not{p}_2 - \not{k}_1 - m - (\not{p}_2 - m)\right) \frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \not{\varepsilon}_{\lambda_2 k_2} \Bigg) u_{\alpha_1 p_1}. \end{aligned}$$

If we now use $(\not{p}_1 - m) u_{\alpha_1 p_1} = 0$ and $\bar{u}_{\alpha_2 p_2}(\not{p}_2 - m) = 0$, what is left cancels the denominator, since we now from momentum conservation that $p_2 - k_1 = p_1 - k_2$. Finally, we arrive at zero:

$$i\mathcal{M}(\varepsilon_{\lambda_1 k_1} \to k_1) = -ig^2 \bar{u}_{\alpha_2 p_2} (\not{\varepsilon}_{\lambda_2 k_2} - \not{\varepsilon}_{\lambda_2 k_2}) u_{\alpha_1 p_1} = 0.$$

The same will also work for $\mathcal{M}(\varepsilon_{\lambda_2 k_2} \to k_2)$. Thus, we can use

$$\sum_{\lambda=1,2} \varepsilon^\mu_{\lambda k} \varepsilon^\nu_{\lambda k} = -\eta^{\mu\nu}.$$

Thus, we get

$$\begin{aligned} \overline{|\mathcal{M}|^2} = \frac{g^4}{4} \sum_{\lambda_1, \lambda_2} & \varepsilon^\mu_{\lambda_2 k_2} \varepsilon^\nu_{\lambda_1 k_1} \varepsilon^\sigma_{\lambda_1 k_1} \varepsilon^\kappa_{\lambda_2 k_2} \\ & \operatorname{Tr}\,(\not{p}_2 + m) \left(\gamma_\mu \frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \gamma_\nu + \gamma_\nu \frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \gamma_\mu \right) \\ & (\not{p}_1 + m) \left(\gamma_\sigma \frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \gamma_\kappa + \gamma_\kappa \frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \gamma_\sigma \right) \\ = \frac{g^4}{4} \eta^{\nu\sigma} \eta^{\mu\kappa} & \operatorname{Tr}\,(\not{p}_2 + m) \left(\gamma_\mu \frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \gamma_\nu + \gamma_\nu \frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \gamma_\mu \right) \\ & (\not{p}_1 + m) \left(\gamma_\sigma \frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \gamma_\kappa + \gamma_\kappa \frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \gamma_\sigma \right) \\ = \frac{g^4}{4} & \operatorname{Tr}\,(\not{p}_2 + m) \left(\gamma_\mu \frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \gamma_\nu + \gamma_\nu \frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \gamma_\mu \right) \\ & (\not{p}_1 + m) \left(\gamma^\nu \frac{1}{(\not{p}_1 + \not{k}_1) - m + i\epsilon} \gamma^\mu + \gamma^\mu \frac{1}{(\not{p}_1 - \not{k}_2) - m + i\epsilon} \gamma^\nu \right) \end{aligned}$$

10.4 Bringing the γ-Matrices into the Numerator

10.4.1 Bringing the γ-Matrices into the Numerator

In (>5.5.3) we derived the identity

$$\frac{1}{\not{p} - m + i\epsilon} = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}.$$

In our case, the p is actually a sum (or difference) of an electron and a photon momentum $p \pm k$. Consider

$$(p \pm k)^2 - m^2 = p^2 \pm 2p \cdot k + k^2 - m^2 = \pm 2p \cdot k,$$

using $p^2 = m^2$ and $k^2 = 0$. Thus, we get

$$\overline{|\mathcal{M}|^2} = \frac{g^4}{4} \mathrm{Tr}\,(\not{p}_2 + m)\left(\gamma_\mu \frac{\not{p}_1 + \not{k}_1 + m}{2p_1 \cdot k_1 + i\epsilon}\gamma_\nu + \gamma_\nu \frac{\not{p}_1 - \not{k}_2 + m}{-2p_1 \cdot k_2 + i\epsilon}\gamma_\mu\right)$$
$$(\not{p}_1 + m)\left(\gamma^\nu \frac{\not{p}_1 + \not{k}_1 + m}{2p_1 \cdot k_1 + i\epsilon}\gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2 + m}{-2p_1 \cdot k_2 + i\epsilon}\gamma^\nu\right).$$

We can simplify things further. Note, that

$$\begin{aligned}
&\not{a}\gamma^\nu = a_\mu \gamma^\mu \gamma^\nu = a_\mu(2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu) = -\gamma^\nu \not{a} + 2a^\nu,\\
&\gamma^\nu \not{a} = \gamma^\nu \gamma^\mu a_\mu = (2\eta^{\nu\mu} - \gamma^\mu\gamma^\nu)a_\mu = -\not{a}\gamma^\nu + 2a^\nu,\\
&(-\not{p} + m)(\not{p} + m) = -p^2 + m^2 = 0.
\end{aligned}$$

If we use those identities, we find

$$\begin{aligned}
(\not{p}_1 \pm \not{k} + m)\gamma^\nu(\not{p}_1 + m) &= (-\gamma^\nu \not{p}_1 + 2p_1^\nu \pm \not{k}\gamma^\nu + m\gamma^\nu)(\not{p}_1 + m)\\
&= (\gamma^\nu(-\not{p}_1 + m) + 2p_1^\nu \pm \not{k}\gamma^\nu)(\not{p}_1 + m) = (2p_1^\nu \pm \not{k}\gamma^\nu)(\not{p}_1 + m),
\end{aligned}$$

$$\begin{aligned}
(\not{p}_1 + m)\gamma^\nu(\not{p}_1 \pm \not{k} + m) &= (\not{p}_1 + m)(-\not{p}_1\gamma^\nu + 2p_1^\nu \pm \gamma^\nu \not{k} + \gamma^\nu m)\\
&= (\not{p}_1 + m)\big((-\not{p}_1 + m)\gamma^\nu + 2p_1^\nu \pm \gamma^\nu \not{k}\big) = (\not{p}_1 + m)(2p_1^\nu \pm \gamma^\nu \not{k}).
\end{aligned}$$

Using those expressions, the sum over our amplitude becomes

$$\overline{|\mathcal{M}|^2} = \frac{g^4}{4} \mathrm{Tr}\,(\not{p}_2 + m)\left(\gamma_\mu \frac{2p_{1\nu} + \not{k}_1\gamma_\nu}{2p_1 \cdot k_1 + i\epsilon} + \gamma_\nu \frac{2p_{1\mu} - \not{k}_2\gamma_\mu}{-2p_1 \cdot k_2 + i\epsilon}\right)$$
$$(\not{p}_1 + m)\left(\frac{2p_1^\nu + \gamma^\nu \not{k}_1}{2p_1 \cdot k_1 + i\epsilon}\gamma^\mu + \frac{2p_1^\mu - \gamma^\mu \not{k}_2}{-2p_1 \cdot k_2 + i\epsilon}\gamma^\nu\right).$$

10.5 Get Rid of the γ-Matrices

10.5.1 Get Rid of the γ-Matrices – First Term

So far, our formula looks like

$$\overline{|\mathcal{M}|^2} \sim \mathrm{Tr}\,\cdots\left(\frac{\cdots}{2p_1 \cdot k_1 + i\epsilon} + \frac{\cdots}{-2p_1 \cdot k_2 + i\epsilon}\right)\cdots\left(\frac{\cdots}{2p_1 \cdot k_1 + i\epsilon} + \frac{\cdots}{-2p_1 \cdot k_2 + i\epsilon}\right).$$

If we multiply out those brackets, we get four terms. Let's consider the first term, which is $\sim 1/(2p_1 \cdot k_1 + i\epsilon)^2$:

$$\begin{aligned}
&\frac{g^4}{4} \mathrm{Tr}\,(\not{p}_2 + m)\left(\gamma_\mu \frac{2p_{1\nu} + \not{k}_1\gamma_\nu}{2p_1 \cdot k_1 + i\epsilon}\right)(\not{p}_1 + m)\left(\frac{2p_1^\nu + \gamma^\nu \not{k}_1}{2p_1 \cdot k_1 + i\epsilon}\gamma^\mu\right)\\
&\quad = \frac{g^4}{4(2p_1 \cdot k_1 + i\epsilon)^2} \mathrm{Tr}\,\gamma^\mu(\not{p}_2 + m)\gamma_\mu(2p_{1\nu} + \not{k}_1\gamma_\nu)(\not{p}_1 + m)(2p_1^\nu + \gamma^\nu \not{k}_1)\\
&\quad = \frac{g^4}{4(2p_1 \cdot k_1 + i\epsilon)^2} \mathrm{Tr}\,\underbrace{\gamma^\mu(\not{p}_2 + m)\gamma_\mu}_{=:A}\\
&\qquad\qquad \Big(\underbrace{2p_{1\nu}(\not{p}_1 + m)2p_1^\nu}_{=4m^2(\not{p}_1 + m)} + \underbrace{2p_{1\nu}(\not{p}_1 + m)\gamma^\nu \not{k}_1 + \not{k}_1\gamma_\nu(\not{p}_1 + m)2p_1^\nu}_{=:B} + \underbrace{\not{k}_1\gamma_\nu(\not{p}_1 + m)\gamma^\nu \not{k}_1}_{=:C}\Big).
\end{aligned}$$

Using $\gamma^\mu\gamma_\mu = 4$ and $\gamma^\mu \not{a}\gamma_\mu = -2\not{a}$ yields

$$A = \gamma^\mu(\not{p}_2 + m)\gamma_\mu = -2\not{p}_2 + 4m.$$

Using $(\not{p}_1 - m)(\not{p}_1 + m) = p_1^2 - m^2 = 0 \Leftrightarrow \not{p}_1(\not{p}_1 + m) = (\not{p}_1 + m)\not{p}_1 = m(\not{p}_1 + m)$ yields

$$
\begin{aligned}
B &= 2(\not{p}_1+m)\not{p}_1\not{k}_1 + 2\not{k}_1\not{p}_1(\not{p}_1+m) = 2m\big((\not{p}_1+m)\not{k}_1 + \not{k}_1(\not{p}_1+m)\big)\\
&= 2m(2m\not{k}_1 + \not{p}_1\not{k}_1 + \not{k}_1\not{p}_1) = 2m(2m\not{k}_1 + \{\not{p}_1,\not{k}_1\}) = 2m(2m\not{k}_1 + 2p_1\cdot k_1)\\
&= 4m(m\not{k}_1 + p_1\cdot k_1).
\end{aligned}
$$

Using the result of term A as well as $\not{k}_1^2 = k_1^2 = 0$ yields

$$
\begin{aligned}
C &= \not{k}_1\gamma_\nu(\not{p}_1+m)\gamma^\nu\not{k}_1 = \not{k}_1(-2\not{p}_1+4m)\not{k}_1 = -2\not{k}_1\not{p}_1\not{k}_1 = -2(\{\not{k}_1,\not{p}_1\} - \not{p}_1\not{k}_1)\not{k}_1\\
&= -4k_1\cdot p_1\not{k}_1.
\end{aligned}
$$

If we plug our results for A, B, C in again, we arrive at the expression

$$
\begin{aligned}
&\frac{g^4}{4(2p_1\cdot k_1+i\epsilon)^2}\operatorname{Tr}(-2\not{p}_2+4m)(4m^2(\not{p}_1+m)+4m(m\not{k}_1+p_1\cdot k_1)-4k_1\cdot p_1\not{k}_1)\\
&\quad= \frac{2g^4}{(2p_1\cdot k_1+i\epsilon)^2}\operatorname{Tr}(-\not{p}_2+2m)(m^2(\not{p}_1+m)+m(m\not{k}_1+p_1\cdot k_1)-k_1\cdot p_1\not{k}_1).
\end{aligned}
$$

A general rule for γ-matrices is that traces over an odd number of them vanishes. This reduces our expression, after multiplying out the $(-\not{p}_2+2m)$-bracket, to

$$
\frac{2g^4}{(2p_1\cdot k_1+i\epsilon)^2}\operatorname{Tr}\big(-\not{p}_2(m^2\not{p}_1+m^2\not{k}_1-k_1\cdot p_1\not{k}_1)+2m(m^3+mp_1\cdot k_1)\big).
$$

Now, we can use

$$
\begin{aligned}
&\not{a}\,\not{b} = a_\mu\gamma^\mu b_\nu\gamma^\nu = a_\mu b_\nu(\{\gamma^\mu,\gamma^\nu\}-\gamma^\nu\gamma^\mu) = 2a\cdot b - \not{b}\,\not{a},\\
&\operatorname{Tr}\not{a}\,\not{b} = \frac{1}{2}(\operatorname{Tr}\not{a}\,\not{b}+\operatorname{Tr}\not{b}\,\not{a}) = \frac{1}{2}(\operatorname{Tr}(2a\cdot b-\not{b}\,\not{a})+\operatorname{Tr}(2a\cdot b-\not{a}\,\not{b})) = \frac{1}{2}(16a\cdot b-2\operatorname{Tr}\not{a}\,\not{b})\\
&\qquad= 8a\cdot b-\operatorname{Tr}\not{a}\,\not{b}\\
&\Leftrightarrow\quad \operatorname{Tr}\not{a}\,\not{b} = 4a\cdot b
\end{aligned}
$$

and we get, taking the limit $\epsilon\to 0$,[1]

$$
\frac{8g^4}{(2p_1\cdot k_1)^2}\big(-m^2p_2\cdot p_1-m^2p_2\cdot k_1+(k_1\cdot p_1)(p_2\cdot k_1)+2m(m^3+mp_1\cdot k_1)\big).
$$

10.5.2 Get Rid of the γ-Matrices – Second Term

The second term of the expression in the beginning of (>10.5.1) is the term proportional to $1/\big((2p_1\cdot k_1+i\epsilon)(-2p_1\cdot k_2+i\epsilon)\big)$-term, which reads

$$
\begin{aligned}
&\frac{g^4}{4}\operatorname{Tr}\,(\not{p}_2+m)\left(\gamma_\mu\frac{2p_{1\nu}+\not{k}_1\gamma_\nu}{2p_1\cdot k_1+i\epsilon}\right)(\not{p}_1+m)\left(\frac{2p_1^\mu-\gamma^\mu\not{k}_2}{-2p_1\cdot k_2+i\epsilon}\gamma^\nu\right)\\
&\quad= \frac{g^4}{4(2p_1\cdot k_1+i\epsilon)(-2p_1\cdot k_2+i\epsilon)}\operatorname{Tr}\,\underbrace{\big(2p_1^\mu-\gamma^\mu\not{k}_2\big)\gamma^\nu(\not{p}_2+m)\gamma_\mu}_{=:A}(2p_{1\nu}+\not{k}_1\gamma_\nu)(\not{p}_1+m).
\end{aligned}
$$

We multiply out A and use $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu$ and $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda}$ for the last two terms:

$$
\begin{aligned}
A &= 2p_1^\mu\gamma^\nu\not{p}_2\gamma_\mu + 2p_1^\mu\gamma^\nu m\gamma_\mu - \gamma^\mu\not{k}_2\gamma^\nu\not{p}_2\gamma_\mu - \gamma^\mu\not{k}_2\gamma^\nu m\gamma_\mu\\
&= 2\gamma^\nu\not{p}_2\not{p}_1 + 2m\gamma^\nu\not{p}_1 + 2\not{p}_2\gamma^\nu\not{k}_2 - 4mk_2^\nu.
\end{aligned}
$$

A contains now four terms and each of them is multiplied with $(2p_{1\nu}+\not{k}_1\gamma_\nu)(\not{p}_1+m)$ in the trace. Let's examine those four terms together with the factor $(2p_{1\nu}+k_1\gamma_\nu)(p_1+m)$ individually. Note, that

[1] Obviously, taking the limit $\epsilon\to 0$ is just the same as putting $\epsilon = 0$, so the question may arise, what is the effect of the ϵ, if the limit is taken that trivially. The answer is that in this example we indeed could have ignored the ϵ in the first place, but this limit is taken much less trivially when the diagram has loops.

all traces over a odd number of γ-matrices vanish automatically. We also again use the identities we just used before:

$$
\begin{aligned}
&\operatorname{Tr} 2\gamma^\nu \not{p}_2 \not{p}_1 (2p_{1\nu} + \not{k}_1\gamma_\nu)(\not{p}_1 + m) = 2\operatorname{Tr}\gamma^\nu \not{p}_2\not{p}_1(2p_{1\nu}\not{p}_1 + \not{k}_1\gamma_\nu\not{p}_1) \\
&\quad = 4\underbrace{\operatorname{Tr}\not{p}_1\not{p}_2\not{p}_1\not{p}_1}_{=4m^2 p_1\cdot p_2} + 2\operatorname{Tr}(\underbrace{\gamma^\nu\not{p}_2\not{p}_1\not{k}_1\gamma_\nu}_{=-2\not{k}_1\not{p}_1\not{p}_2}\not{p}_1) = 16m^2 p_1\cdot p_2 - 4\operatorname{Tr}\not{k}_1\not{p}_1\not{p}_2\not{p}_1 \\
&\quad = 16(m^2 p_1\cdot p_2 - 2(k_1\cdot p_1)(p_1\cdot p_2) + m^2 k_1\cdot p_2),
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{Tr} 2m\gamma^\nu\not{p}_1(2p_{1\nu} + \not{k}_1\gamma_\nu)(\not{p}_1 + m) = 2m^2\operatorname{Tr}\gamma^\nu\not{p}_1(2p_{1\nu} + \not{k}_1\gamma_\nu) \\
&\quad = 4m^2 p_{1\nu}\underbrace{\operatorname{Tr}\gamma^\nu\not{p}_1}_{=4p_1^\nu} + 2m^2\operatorname{Tr}\underbrace{\gamma^\nu\not{p}_1\not{k}_1\gamma_\nu}_{=4p_1\cdot k_1\mathbb{I}} = 16m^4 + 32m^2 p_1\cdot k_1,
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{Tr} 2\not{p}_2\gamma^\nu\not{k}_2(2p_{1\nu} + \not{k}_1\gamma_\nu)(\not{p}_1 + m) = \operatorname{Tr} 2\not{p}_2\gamma^\nu\not{k}_2(2p_{1\nu}\not{p}_1 + \not{k}_1\gamma_\nu\not{p}_1) \\
&\quad = \underbrace{4\operatorname{Tr}\not{p}_2\not{p}_1\not{k}_2\not{p}_1}_{=4((p_1\cdot p_2)(k_2\cdot p_1) - m^2 p_2\cdot k_2 + (p_1\cdot p_2)(p_1\cdot k_2))} + 2\operatorname{Tr}\not{p}_2\underbrace{\gamma^\nu\not{k}_2\not{k}_1\gamma_\nu}_{=4k_1\cdot k_2}\not{p}_1 \\
&\quad = 16(2(p_1\cdot p_2)(k_2\cdot p_1) - m^2 p_2\cdot k_2) + 8(k_1\cdot k_2)\underbrace{\operatorname{Tr}\not{p}_2\not{p}_1}_{=4p_1\cdot p_2} \\
&\quad = 16(2(p_1\cdot p_2)(k_2\cdot p_1) - m^2 p_2\cdot k_2) + 32(p_1\cdot p_2)(k_1\cdot k_2),
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{Tr}\big(-4mk_2^\nu(2p_{1\nu} + \not{k}_1\gamma_\nu)(\not{p}_1 + m)\big) = -4m^2 k_2^\nu\operatorname{Tr}(2p_{1\nu} + \not{k}_1\gamma_\nu) = -4m^2 k_2^\nu(8p_{1\nu} + 4k_{1\nu}) \\
&\quad = -32m^2 p_1\cdot k_2 - 16m^2 k_1\cdot k_2.
\end{aligned}
$$

Plugging the sum of those four terms back into our "second term", we find (again, in the limit $\epsilon \to 0$)

$$
\begin{aligned}
&\frac{g^4}{4(2p_1\cdot k_1)(-2p_1\cdot k_2)}(\text{sum of the four trace terms}) \\
&\quad = \frac{g^4}{4(2p_1\cdot k_1)(-2p_1\cdot k_2)}(16m^2 p_1\cdot p_2 - 32(k_1\cdot p_1)(p_1\cdot p_2) + 16m^2 k_1\cdot p_2 + 16m^4 \\
&\quad + 32m^2 p_1\cdot k_1 + 32(p_1\cdot p_2)(k_2\cdot p_1) - 16m^2 p_2\cdot k_2 + 32(p_1\cdot p_2)(k_1\cdot k_2) - 32m^2 p_1\cdot k_2 \\
&\quad - 16m^2 k_1\cdot k_2).
\end{aligned}
$$

10.5.3 Get Rid of the γ-Matrices – Third Term

The third term of the expression in the beginning of (>10.5.1) is the term proportional to $1/\big((-2p_1\cdot k_2 + i\epsilon)(2p_1\cdot k_1 + i\epsilon)\big)$-term, which reads

$$
\frac{g^4}{4}\operatorname{Tr}(\not{p}_2 + m)\left(\gamma_\nu\frac{2p_{1\mu} - \not{k}_2\gamma_\mu}{-2p_1\cdot k_2 + i\epsilon}\right)(\not{p}_1 + m)\left(\frac{2p_1^\nu + \gamma^\nu\not{k}_1}{2p_1\cdot k_1 + i\epsilon}\gamma^\mu\right),
$$

which is the same as the second term for $k_1 \leftrightarrow -k_2$ interchanged. Thus, we can copy its result and interchange the k'2 and we get

$$
\begin{aligned}
&\frac{g^4}{4(-2p_1\cdot k_2)(2p_1\cdot k_1)}(16m^2 p_1\cdot p_2 + 32(k_2\cdot p_1)(p_1\cdot p_2) - 16m^2 k_2\cdot p_2 + 16m^4 \\
&\quad - 32m^2 p_1\cdot k_2 - 32(p_1\cdot p_2)(k_1\cdot p_1) + 16m^2 p_2\cdot k_1 + 32(p_1\cdot p_2)(k_1\cdot k_2) + 32m^2 p_1\cdot k_1 \\
&\quad - 16m^2 k_1\cdot k_2).
\end{aligned}
$$

10.5.4 Get Rid of the γ-Matrices – Fourth Term

The fourth term of the expression in the beginning of (>10.5.1) is the term proportional to $1/\big((-2p_1\cdot k_2 + i\epsilon)(-2p_1\cdot k_2 + i\epsilon)\big)$-term, which reads

$$
\frac{g^4}{4}\operatorname{Tr}(\not{p}_2 + m)\left(\gamma_\nu\frac{2p_{1\mu} - \not{k}_2\gamma_\mu}{-2p_1\cdot k_2 + i\epsilon}\right)(\not{p}_1 + m)\left(\frac{2p_1^\mu - \gamma^\mu\not{k}_2}{-2p_1\cdot k_2 + i\epsilon}\right),
$$

which is the same as the first term if we replace $k_1 \leftrightarrow -k_2$. Thus, we can copy its result and interchange the k'2 and we get

$$\frac{8g^4}{(2p_1\cdot k_2)^2}\big(-m^2p_2\cdot p_1+m^2p_2\cdot k_2+(k_2\cdot p_1)(p_2\cdot k_2)+2m(m^3-mp_1\cdot k_2)\big).$$

10.6 Mandelstam Variables

10.6.1 Sum of Mandelstam Variables

Consider a process of four particles with arbitrary masses, such that $p_i^2=m_{p_i}^2$ and $k_i^2=m_{k_i}^2$. Then, the sum of the three Mandelstam variables reads

$$\begin{aligned}s+t+u&=(p_1+k_1)^2+(p_1-p_2)^2+(p_1-k_2)^2=3p_1^2+2p_1\cdot(k_1-p_2-k_2)+k_1^2+p_2^2+k_2^2\\&=3p_1^2-2p_1^2+k_1^2+p_2^2+k_2^2=m_{p_1^2}+m_{p_2^2}+m_{k_1^2}+m_{k_2^2}.\end{aligned}$$

where we used the momentum conservation law $p_1+k_1=p_2+k_2\Leftrightarrow k_1-p_2-k_2=-p_1$.

10.6.2 Dot Products in Terms of Mandelstam Variables

We will now define the so-called Mandelstam variables

$$\begin{aligned}&s:=(p_1+k_1)^2=(p_2+k_2)^2,\\&t:=(p_1-p_2)^2=(k_1-k_2)^2,\\&u:=(p_1-k_2)^2=(p_2-k_1)^2,\end{aligned}$$

with which we can substitute the following dot products:

$$\begin{aligned}&2p_1\cdot k_1=s-p_1^2-k_1^2=s-m^2=:S,\\&2p_2\cdot k_2=s-p_2^2-k_2^2=s-m^2=S,\\&2p_1\cdot k_2=-u+p_1^2+k_2^2=m^2-u=:-U,\\&2p_2\cdot k_1=-u+p_2^2+k_2^2=m^2-u=-U.\\&2p_1\cdot p_2=-t+p_1^2+p_2^2=2m^2-t=2m^2-(2m^2-u-s)=U+S+2m^2,\\&2k_1\cdot k_2=-t+k_1^2+k_2^2=-t=s+u-2m^2=S+U,\end{aligned}$$

To express t in terms of u and s we used $s+t+u=2m^2$, we is the sum of the squared masses of all particles.

10.6.3 Squared Matrix Element in Terms of Mandelstam Variables

In 10.5 we considered four terms separately. Let's now takes those four terms, again separately, and substitute the Mandelstam variables (actually, we are going to substitute S and U).

First term:

$$\begin{aligned}&\frac{8g^4}{(2p_1\cdot k_1)^2}(-m^2p_1\cdot p_2-m^2p_2\cdot k_1+(p_1\cdot k_1)(p_2\cdot k_1)+2m^4+2m^2p_1\cdot k_1)\\&\quad=\frac{2g^4}{S^2}(-2m^2(U+S+2m^2)+2m^2U-SU+8m^4+4m^2S)=\frac{2g^4}{S^2}(4m^4+2m^2S-SU)\\&\quad=2g^4\left(\frac{4m^4}{S^2}+\frac{2m^2}{S}-\frac{U}{S}\right)\end{aligned}$$

Second term:

$$\frac{g^4}{4(2p_1\cdot k_1)(-2p_1\cdot k_2)}(16m^2p_1\cdot p_2-32(p_1\cdot k_1)(p_1\cdot p_2)+16m^2p_2\cdot k_1+16m^4+32m^2p_1\cdot k_1+32(p_1\cdot p_2)(p_1\cdot k_2)-16m^2p_2\cdot k_2+32(p_1\cdot p_2)(k_1\cdot k_2)-32m^2p_1\cdot k_2-16m^2k_1\cdot k_2)$$
$$=\frac{2g^4}{SU}\left(m^2(U+S+2m^2)-S(U+S+2m^2)-m^2U+2m^4+2m^2S-(U+S+2m^2)U-m^2S+(U+S+2m^2)(S+U)+2m^2U-m^2(S+U)\right)=\frac{2g^4}{SU}(4m^4+m^2U+m^2S)$$
$$=2g^4\left(\frac{4m^4}{SU}+\frac{m^2}{S}+\frac{m^2}{U}\right)$$

Third Term:

$$\frac{g^4}{4(-2p_1\cdot k_2)(2p_1\cdot k_1)}(16m^2p_1\cdot p_2+32(p_1\cdot k_2)(p_1\cdot p_2)-16m^2p_2\cdot k_2+16m^4-32m^2p_1\cdot k_2-32(p_1\cdot p_2)(p_1\cdot k_1)+16m^2p_2\cdot k_1+32(p_1\cdot p_2)(k_1\cdot k_2)+32m^2p_1\cdot k_1-16m^2k_1\cdot k_2)$$
$$=\frac{2g^4}{US}(m^2(U+S+2m^2)-U(U+S+2m^2)-m^2S+2m^4+2m^2U-(U+S+2m^2)S-m^2U+(U+S+2m^2)(S+U)+2m^2S-m^2S-m^2U)=\frac{2g^4}{US}(4m^4+m^2U+m^2S)$$
$$=2g^4\left(\frac{4m^4}{SU}+\frac{m^2}{S}+\frac{m^2}{U}\right)$$

Fourth term:

$$\frac{8g^4}{(2p_1\cdot k_2)^2}(-m^2p_2\cdot p_1+m^2p_2\cdot k_2+(p_1\cdot k_2)(p_2\cdot k_2)+2m^4-2m^2p_1\cdot k_2)$$
$$=\frac{2g^4}{U^2}(-2m^2(U+S+2m^2)+2m^2S-US+8m^4+4m^2U)=\frac{2g^4}{U^2}(4m^4+2m^2U-US)$$
$$=2g^4\left(\frac{4m^4}{U^2}+\frac{2m^2}{U}-\frac{S}{U}\right).$$

Summing all four terms up yields

$$\overline{|\mathcal{M}|^2}=2g^4\left(\frac{4m^4}{S^2}+\frac{2m^2}{S}-\frac{U}{S}\right)+2g^4\left(\frac{4m^4}{SU}+\frac{m^2}{S}+\frac{m^2}{U}\right)+2g^4\left(\frac{4m^4}{SU}+\frac{m^2}{S}+\frac{m^2}{U}\right)+2g^4\left(\frac{4m^4}{U^2}+\frac{2m^2}{U}-\frac{S}{U}\right)=2g^4\left(4m^4\left(\frac{1}{S}+\frac{1}{U}\right)^2+4m^2\left(\frac{1}{S}+\frac{1}{U}\right)-\frac{U}{S}-\frac{S}{U}\right)$$

10.6.4 Mandelstam Variables in the Center of Mass Frame

We consider the center of mass frame and choose our coordinates such that $\vec{k}_1$ points along the z-axis:

$$k_1^\mu=k_1^0(1,0,0,1),\quad p_1^\mu=p_1^0(1,0,0,-\beta).$$

Since we are in the center of mass frame, we have $k_1^0=k_2^0$ and $p_1^0=p_2^0$. The only thing which can change for the outgoing particles with respect to the incoming ones is the direction, in which they go to. We can specify this direction in terms of a single scattering θ, as the amplitude should be symmetric with respect to a polar angle φ. Thus, we can write

$$k_2^\mu=k_1^0(1,\sin\theta,0,\cos\theta),\quad p_2^\mu=p_1^0(1,-\beta\sin\theta,0,-\beta\cos\theta),$$

such that we still have $k_2^2=k_1^2=0$ and $p_2^2=p_1^2$. If we now use the dependencies

$$\vec{p}_1 + \vec{k}_1 = (k_1^0 - p_1^0\beta)\hat{z} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \beta = \frac{k_1^0}{p_1^0},$$

$$s = (p_1 + k_1)^2 = (p_1^0 + k_1^0)^2,$$

where we used for the last equation that in the center of mass frame we have $\vec{p}_1 + \vec{k}_1 = 0$, we find

$$s = (p_1 + k_1)^2 = m^2 + 2p_1 \cdot k_1 = m^2 + 2k_1^0 p_1^0(1 + \beta) = m^2 + 2k_1^0(p_1^0 + k_1^0) = m^2 + 2k_1^0\sqrt{s}$$

$$\Leftrightarrow \quad k_1^0 = \frac{s - m^2}{2\sqrt{s}} \quad \Leftrightarrow \quad p_1^0 = \sqrt{s} - k_1^0 = \frac{s + m^2}{2\sqrt{s}}$$

and thus

$$u = (p_1 - k_2)^2 = m^2 - 2p_1 \cdot k_2 = m^2 - 2p_1^0 k_1^0(1 + \beta\cos\theta) = m^2 - \frac{s^2 - m^4}{2s}(1 + \beta\cos\theta).$$

11 The Optical Theorem and the Ward-Takahashi Identity

11.1 The Principle of the Optical Theorem

11.1.1 Equation of the T-Matrix

If we plug $S = 1 + iT$ into $S^\dagger S = 1$, we find

$$1 = S^\dagger S = \left(1 - iT^\dagger\right)(1 + iT) = 1 + iT - iT^\dagger + T^\dagger T \qquad \Leftrightarrow \qquad -i\left(T - T^\dagger\right) = T^\dagger T.$$

11.1.2 Derivation of the Standard Form of the Optical Theorem

If we put a matrix element with two arbitrary multi-particle states $|\{p_i\}\rangle$, $|\{p_i'\}\rangle$ around both sides of the equation $-i\left(T - T^\dagger\right) = T^\dagger T$, we find

$$-i\left(\langle\{p_i'\}|T|\{p_i\}\rangle - \left\langle\{p_i'\}\middle|T^\dagger\middle|\{p_i\}\right\rangle\right) = \left\langle\{p_i'\}\middle|T^\dagger T\middle|\{p_i\}\right\rangle.$$

In section (>8.2.2), we defined a matrix element $\mathcal{M}$ as

$$(2\pi)^4\delta(p - q) \cdot i\mathcal{M}_{\{p_i\},\{q_i\}_n} \coloneqq i\langle\{q_i\}_n|T|\{p_i\}\rangle$$

$$\Rightarrow \qquad \left\langle\{p_i\}\middle|T^\dagger\middle|\{q_i\}_n\right\rangle = \langle\{q_i\}_n|T|\{p_i\}\rangle^\dagger = (2\pi)^4\delta(p - q) \cdot \mathcal{M}^*_{\{p_i\},\{q_i\}_n}.$$

Thus, the left-hand side of the equation reads

$$-i\left(\langle\{p_i'\}|T|\{p_i\}\rangle - \left\langle\{p_i'\}\middle|T^\dagger\middle|\{p_i\}\right\rangle\right) = -i\left(\mathcal{M}_{\{p_i\},\{p_i'\}} - \mathcal{M}^*_{\{p_i'\},\{p_i\}}\right) \cdot (2\pi)^4\delta(p - p').$$

To write the right-hand side in terms of matrix elements $\mathcal{M}$, we neeed to plug in a complete set of intermediate states,

$$\left\langle\{p_i'\}\middle|T^\dagger T\middle|\{p_i\}\right\rangle = \sum_{n=1}^{\infty} \int \{d\tilde{q}_i\}_n \left\langle\{p_i'\}\middle|T^\dagger\middle|\{q_i\}_n\right\rangle\langle\{q_i\}_n|T|\{p_i\}\rangle,$$

where $|\{q_i\}_n\rangle$ is an n-particle intermediate state. We also used $\{d^3\tilde{q}_i\}_n \coloneqq \prod_{i=1}^n d^3\tilde{q}_i$ as a short-hand notation. Now we can insert the matrix elements $\mathcal{M}$:

$$\left\langle\{p_i'\}\middle|T^\dagger T\middle|\{p_i\}\right\rangle = \sum_{n=1}^{\infty} \int \{d\tilde{q}_i\}_n \, \mathcal{M}^*_{\{p_i'\},\{q_i\}_n}\mathcal{M}_{\{p_i\},\{q_i\}_n} (2\pi)^4\delta(p' - q)(2\pi)^4\delta(p - q).$$

Since the second δ-function ensures that $p = q$, we can set $q = p$ in the first δ-funtion. This δ-function $(2\pi)^4\delta(p' - p)$ also appears in the result for the left-hand side and therefore drops off. We are left with

$$-i\left(\mathcal{M}_{\{p_i\},\{p_i'\}} - \mathcal{M}^*_{\{p_i'\},\{p_i\}}\right) = \sum_{n=1}^{\infty} \int \{d\tilde{q}_i\}_n \, (2\pi)^4\delta(p - q) \, \mathcal{M}^*_{\{p_i'\},\{q_i\}_n}\mathcal{M}_{\{p_i\},\{q_i\}_n}.$$

11.1.3 Special Case of Forward Scattering

Let's consider the case of forward scattering of two particles, i. e. $\{p_i\} = \{p_i'\} = \{p_1, p_2\}$. Since it always holds that $A - A^* = 2i \operatorname{Im} A$, we find

$$2\operatorname{Im}\mathcal{M}(p_1, p_2 \to p_1, p_2) = \sum_{n=1}^{\infty} \int \underbrace{\{d\tilde{q}_i\}_n \, (2\pi)^4\delta(p - q)}_{= d\phi_n} \, |\mathcal{M}(p_1, p_2 \to \{q_i\}_n)|^2.$$

Recall from section 9.2 the formula

$$|\mathcal{M}(p_1, p_2 \to \{q_i\}_n)|^2 \, d\phi_n = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}\, d\sigma,$$

where $d\phi_n = \{d\tilde{q}_i\}_n \cdot (2\pi)^4 \delta(q - p)$. Thus, we find

$$2\,\mathrm{Im}\,\mathcal{M}(p_1, p_2 \to p_1, p_2) = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}\underbrace{\sum_{n=1}^{\infty} \int d\sigma(p_1, p_2 \to \{q_i\}_n)}_{=\sigma_{\text{tot}}(p_1, p_2 \to \text{anything})}.$$

11.1.4 Optical Theorem in Centre of Mass Frame

In the centre of mass frame, it is

$$p_1 = \begin{pmatrix} E_1 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \sqrt{\vec{p}^2 + m_1^2} \\ \vec{p} \end{pmatrix}, \qquad p_2 = \begin{pmatrix} E_2 \\ -\vec{p} \end{pmatrix} = \begin{pmatrix} \sqrt{\vec{p}^2 + m_2^2} \\ -\vec{p} \end{pmatrix}.$$

Thus, the square root in the prefactor can be written as

$$\begin{aligned}
\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} &= \sqrt{(E_1 E_2 + \vec{p}^2)^2 - m_1^2 m_2^2} \\
&= \sqrt{(\vec{p}^2 + m_1^2)(\vec{p}^2 + m_2^2) + 2E_1 E_2 \vec{p}^2 + \vec{p}^4 - m_1^2 m_2^2} = |\vec{p}|\sqrt{2\vec{p}^2 + m_1 + m_2 + 2E_1 E_2} \\
&= |\vec{p}|\sqrt{2p_1 \cdot p_2 + m_1^2 + m_2^2} = |\vec{p}|\sqrt{(p_1 + p_2)^2} = |\vec{p}| E_{\text{cm}},
\end{aligned}$$

where $E_{\text{cm}} = \sqrt{s}$ is the total centre of mass energy and. Thus, in this frame, the optical theorem reads

$$\mathrm{Im}\,\mathcal{M}(p_1, p_2 \to p_1, p_2) = 2|\vec{p}| E_{\text{cm}} \sigma_{\text{tot}}(p_1, p_2 \to \text{anything}).$$

11.2 Branch Cut and Discontinuity

11.2.1 The Matrix Element is Real below the Multiparticle State Threshold

Recall from section 7.3 that an interacting propagator includes all possible N-particle states. The single particle state has an invariant mass $M = m$, where m is the rest mass of the particle, and the invariant masses M of all the other states form a continuum starting from $M > 2m$. Thus, the threshold energy for creating a multiparticle state is $s_0 = 2m$.

If we consider a scattering process with centre of mass energy/invariant mass s, intermediate multiparticle states with *virtual* particles can always be formed, but if those virtual particles should go on-shell, we need an invariant mass $s \geq s_0$. And only then $\mathcal{M}(s)$ receives an imaginary part.

11.2.2 Discontinuity at the Real Axis

We can analytically continue $\mathcal{M}(s)$ and interpret it as a function of a complex parameter $s \in \mathbb{C}$. Since $\mathcal{M}(s)$ is real for $s < s_0$, in this regime we can write

$$\mathcal{M}(s) = \big(\mathcal{M}(s^*)\big)^*,$$

since for any (otherwise) real function of a complex variable $f(z^*) = \big(f(z)\big)^*$ holds. Since both sides of this equation are analytic functions of s, we can analytically continue them over the whole complex plane and the relation must still hold.

We discussed in (>12.2.1) that only for $s \geq s_0$ intermediate states can go on-shell, that is only then the denominators of propagators vanish. Thus, $\mathcal{M}(s)$ is not defined on the real axis for $s \geq s_0$. We say, the complex plane has a *branch cut* at the real axis. But with our analytically continued $\mathcal{M}(s)$ we can still examine its behaviour close to the real axis, that is at $s = s_{\text{R}} \pm i\epsilon$, where $s_{\text{R}} \geq s_0$ and $s_{\text{R}} \in \mathbb{R}$. If we plug in $s = s_{\text{R}} + i\epsilon$ into the equation above and then take the real and imaginary part, we find

$$\mathcal{M}(s_\mathrm{R}+i\epsilon) = \big(\mathcal{M}(s-i\epsilon)\big)^*$$

$$\Leftrightarrow \qquad \operatorname{Re}\mathcal{M}(s_\mathrm{R}+i\epsilon) = \operatorname{Re}\mathcal{M}(s_\mathrm{R}-i\epsilon),$$

$$\operatorname{Im}\mathcal{M}(s_\mathrm{R}+i\epsilon) = -\operatorname{Im}\mathcal{M}(s_\mathrm{R}-i\epsilon),$$

since $\operatorname{Re} A = \operatorname{Re} A^*$ but $\operatorname{Im} A = -\operatorname{Im} A^*$. Focusing on the imaginary part, we see that if we approach the real axis from above, that is $s_\mathrm{R}+i\epsilon$ with $\epsilon \to 0$, we get the imaginary part $\operatorname{Im}\mathcal{M}(s_\mathrm{R}+i\epsilon) = -\operatorname{Im}\mathcal{M}(s_\mathrm{R}-i\epsilon)$, whereas if we approach the real axis from below, that is $s_\mathrm{R}-i\epsilon$ with $\epsilon \to 0$, we of course get $\operatorname{Im}\mathcal{M}(s_\mathrm{R}-i\epsilon)$. The difference is the discontinuity

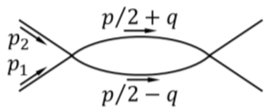
$$\begin{aligned}\operatorname{Disc}\mathcal{M}(s) &\coloneqq i\operatorname{Im}\mathcal{M}(s_\mathrm{R}+i\epsilon) - i\operatorname{Im}\mathcal{M}(s_\mathrm{R}-i\epsilon) = i\operatorname{Im}\mathcal{M}(s_\mathrm{R}+i\epsilon) + i\operatorname{Im}\mathcal{M}(s_\mathrm{R}+i\epsilon)\\ &= 2i\operatorname{Im}\mathcal{M}(s_\mathrm{R}+i\epsilon),\end{aligned}$$

where the limit $\epsilon \to 0$ is implied as usual.

11.3 The Optical Theorem for ϕ^4-Theory

11.3.1 Amplitude with Feynman Rules

The diagram



has – according to the Feynman rules – the amplitude

$$\begin{aligned} i\,\delta\mathcal{M} &= \frac{1}{2}\int d^4\bar{q}\,(-i\lambda)\frac{i}{(p/2+q)^2-m^2+i\epsilon}\frac{i}{(p/2-q)^2-m^2+i\epsilon}(-i\lambda)\\ &= \frac{\lambda^2}{2}\int d^4\bar{q}\,\frac{1}{(p/2+q)^2-m^2+i\epsilon}\frac{1}{(p/2-q)^2-m^2+i\epsilon}.\end{aligned}$$

The factor $1/2$ is the symmetry factor.

11.3.2 Poles of the Amplitude

In the centre of mass frame, where $p = (p^0, 0)$, the q^0-integral has poles at

$$(p/2 \pm q)^2 - m^2 + i\epsilon = 0$$

$$\Leftrightarrow \quad p^2/4 \pm p\cdot q + q^2 - m^2 + i\epsilon = 0$$

$$\Leftrightarrow \quad p^{02}/4 \pm p^0\cdot q^0 + q^{02} - \vec{q}^2 - m^2 + i\epsilon = 0$$

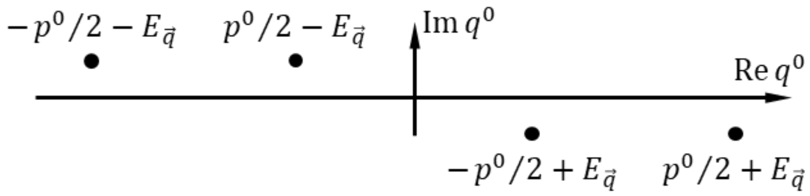
$$\Leftrightarrow \quad (p^0/2 \pm q^0)^2 = E_{\vec{q}}^2 - i\epsilon \qquad \left(\text{using } E_{\vec{q}} \coloneqq \sqrt{\vec{q}^2+m^2}\right)$$

$$\Leftrightarrow \quad p^0/2 \pm q^0 = \pm'\big(E_{\vec{q}} - i\epsilon\big) \qquad \left(\text{using } \sqrt{E_{\vec{q}}^2 - i\epsilon} = E_{\vec{q}} - i\tilde{\epsilon}\right)$$

$$\Leftrightarrow \quad q^0 = \mp p^0/2 \ \pm'\left(E_{\vec{q}} - i\epsilon\right) \eqqcolon q^0_{\mp\pm'},$$

where the sign $\pm'$ can be chosen independently of $\pm$, thus we have four different poles. Note, that the poles $q^0_{\mp\pm'}$ belong to the denominator $(p/2 \pm q)^2 - m^2 + i\epsilon$, thus the dash-less plus-minus sign is swapped.

For $\pm' = +$, the imaginary part of q^0 is negative and for $\pm' = -$ positive. Thus, we have to poles above the real q^0 axis and two below:



Note that the imaginary part of all poles is proportional to ϵ and thus infinitely small. The real part, on the other hand, is less restricted: Since $p^0, E_{\vec{q}} > 0$, the two outer poles will always stay the outer for arbitrary values of p^0 and $E_{\vec{q}}$. Moreover, the real parts of the outer poles and separately also the real parts of the inner poles are symmetrical w. r. t. the imaginary axis.

Since we have also an integral over $\vec{q}$, the energy $E_{\vec{q}}$ can take on arbitrary (positive) values larger than m. $p^0 = p_1^0 + p_2^0$ is fixed to some value larger than $2m$. Thus, the real parts of the two inner poles can be positive or negative, depending on the precise values of p^0 and $E_{\vec{q}}$.

11.3.3 Replace the First Propagator by a Delta-Function

We will close the contour in the lower half plane. Recall residues theorem from the footnote on page 26 and consider the integral

$$\oint dz\, f(z) \frac{1}{z - z_0} = -2\pi i \sum_i \mathrm{res}\left(\frac{f(z)}{z - z_0}, z_i\right),$$

The sum over i or z_i respectively will certainly include $z_i = z_0$, but may also include additional residues of the function f. Let's evaluate the contribution of the residue $z_i = z_0$:

$$\begin{aligned}\oint dz\, f(z) \frac{1}{z - z_0} &= -2\pi i\, \mathrm{res}\left(\frac{f(z)}{z - z_0}, z_0\right) + \cdots = -2\pi i\; (z - z_0) \frac{f(z)}{z - z_0}\bigg|_{z = z_0} + \cdots \\ &= -2\pi i f(z_0) + \cdots = -2\pi i \int dz\, f(z)\, \delta(z - z_0) + \cdots,\end{aligned}$$

where the $+\cdots$ stand for the contribution of the other residues. Thus, picking up the residue z_0 is equivalent to replacing $(z - z_0)^{-1} \to -2\pi i\, \delta(z - z_0)$.

Now consider consider the contribution of the pole q_{-+}^0 (that is, we neglect the "$+\cdots$" here)[1]:

$$\begin{aligned} i\,\delta\mathcal{M} &= \frac{\lambda^2}{2} \int d^4\bar{q}\; \frac{1}{q^0 - q_{++}^0}\; \frac{1}{q^0 - q_{+-}^0}\; \frac{1}{q^0 - q_{-+}^0}\; \frac{1}{q^0 - q_{--}^0} \\ &= -2\pi i \frac{\lambda^2}{2} \int d^4\bar{q}\; \frac{1}{q^0 - q_{++}^0}\; \frac{1}{q^0 - q_{+-}^0}\; \delta(q^0 - q_{-+}^0)\; \frac{1}{q^0 - q_{--}^0} \\ &= -i \frac{\lambda^2}{2} \int d^3\bar{q}\; \frac{1}{q_{-+}^0 - q_{++}^0}\; \frac{1}{q_{-+}^0 - q_{+-}^0}\; \frac{1}{q_{-+}^0 - q_{--}^0} \\ &= -i \frac{\lambda^2}{2} \int d^3\bar{q}\; \frac{1}{-p^0}\; \frac{1}{-p^0 + 2E_{\vec{q}} - i\epsilon}\; \frac{1}{2E_{\vec{q}} - i\epsilon} = -i \frac{\lambda^2}{2} \int d^3\bar{q}\; \frac{1}{2E_{\vec{q}}}\; \frac{1}{p^0\left(p^0 - 2E_{\vec{q}} + i\epsilon\right)} \\ &= -i \frac{\lambda^2}{4} \frac{4\pi}{(2\pi)^3} \int_m^\infty dE_{\vec{q}}\, |\vec{q}| \frac{1}{p^0\left(p^0 - 2E_{\vec{q}}\right)}, \end{aligned}$$

where in the last step we used the substitution

$$E_{\vec{q}} = \sqrt{|\vec{q}|^2 + m^2} \qquad \Longrightarrow \qquad dE_{\vec{q}} = d\sqrt{|\vec{q}|^2 + m^2} = \frac{|\vec{q}|\, d|\vec{q}|}{E_{\vec{q}}}$$

[1] One can show that those will not contribute to the discontinuity.

$$\Rightarrow \qquad d^3q = 4\pi\, |\vec{q}|^2\, d|\vec{q}| = 4\pi\, |\vec{q}|\, E_{\vec{q}}\, dE_{\vec{q}}.$$

Interestingly, it is also possible to get to this result by replacing the whole propagator by the δ-function

$$\frac{1}{(p/2-q)^2-m^2} \to -2\pi i\, \delta((p/2+q)^2-m^2).$$

Obviously, we know from (>11.3.2) that the zeroes of the argument of the δ-function are $q^0_{-\pm'} = -p^0/2 \pm' E_{\vec{q}}$. As before, we will only consider the contribution of q^0_{-+}, as q^0_{--} will not contribute to the discontinuity (a fact, which we will not prove but accept). We find[1]

$$\begin{aligned} i\,\delta\mathcal{M} &\to \frac{\lambda^2}{2}\int d^4\bar{q}\,\left(-2\pi i\, \delta((p/2+q)^2-m^2)\right)\frac{1}{(p/2-q)^2-m^2} \\ &= -2\pi i\frac{\lambda^2}{2}\int d^4\bar{q}\,\frac{1}{|p^0+2q^0_{-+}|}\delta(q^0-q^0_{-+})\frac{1}{p^{02}/4-p^0\cdot q^0+q^{02}-\vec{q}^2-m^2} \\ &= -i\frac{\lambda^2}{2}\int d^3\bar{q}\,\frac{1}{|p^0-p^0+2E_{\vec{q}}|}\frac{1}{(p^0/2-q^0_{-+})^2-E^2_{\vec{q}}} = -i\frac{\lambda^2}{2}\int d^3\bar{q}\,\frac{1}{2E_{\vec{q}}}\frac{1}{\left(p^0-E_{\vec{q}}\right)^2-E^2_{\vec{q}}} \\ &= -i\frac{\lambda^2}{2}\int d^3\bar{q}\,\frac{1}{2E_{\vec{q}}}\frac{1}{p^0\left(p^0-2E_{\vec{q}}\right)} = -i\frac{\lambda^2}{4}\frac{4\pi}{(2\pi)^3}\int_m^\infty dE_{\vec{q}}\,|\vec{q}|\,\frac{1}{p^0\left(p^0-2E_{\vec{q}}\right)}. \end{aligned}$$

The two last steps are precisely the same as for the calculation with residues theorem.

11.3.4 Replace the Second Propagator by a Delta-Function

We are interested into the discontinuity with respect to the Mandelstam variable $s = p^0$ and this discontinuity only exists for $\epsilon \to 0$ (which is, of course, always implied). Obviously now, the integral is convergent for $p^0 < m/2$, such that $\mathcal{M}$ is manifestly real, and divergent for $p^0 > m/2$. If the function $\mathcal{M}(p^0)$ is continued to the whole complex plane, there is a branch cut starting from $p^0 = m/2$ with a discontinuity

$$\operatorname{Disc}\mathcal{M}(p^0) = i\operatorname{Im}\mathcal{M}(p^0+i\epsilon) - i\operatorname{Im}\mathcal{M}(p^0-i\epsilon).$$

To evaluate this discontinuity, we use

$$\frac{1}{x\pm i\epsilon} = \frac{x\mp i\epsilon}{x^2+\epsilon^2} = \frac{x}{x^2+\epsilon^2} \mp \frac{i\epsilon}{x^2+\epsilon^2} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x),$$

where $\mathcal{P}$ denotes the *Cauchy Principal Value*. In our case, this identity reads

$$\frac{1}{p^0-2E_{\vec{q}}\pm i\epsilon} = \mathcal{P}\frac{1}{p^0-2E_{\vec{q}}} \mp i\pi\delta\left(p^0-2E_{\vec{q}}\right)$$

and when we evaluate the discontinuity, only the second term survives:

[1] We use the δ-function identity

$$\delta\left(g(x)\right) = \sum_n \frac{1}{|g'(x_n)|}\delta(x-x_n),$$

where x_n are the zeroes of $g(x)$. In our case, we have $g(q^0) = (p/2+q)^2-m^2$ and hence

$$g'(q^0) = \frac{d}{dq^0}((p/2+q)^2-m^2) = \frac{d}{dq^0}(p^2/4+p\cdot q+q^2-m^2) = p^0+2q^0.$$

$$\begin{aligned}
&\operatorname{Disc}\mathcal{M}(p^0)\\
&\quad= -\frac{\lambda^2}{4}\frac{4\pi}{(2\pi)^3}\int_m^\infty dE_{\vec{q}}\,|\vec{q}|\,\left(i\,\mathrm{Im}\left(\mathcal{P}\frac{1}{x} - i\pi\delta\big(p^0 - 2E_{\vec{q}}\big)\right) - i\,\mathrm{Im}\left(\mathcal{P}\frac{1}{x} + i\pi\delta\big(p^0 - 2E_{\vec{q}}\big)\right)\right)\\
&\quad= i\,\mathrm{Im}\frac{\lambda^2}{4}\frac{4\pi}{(2\pi)^3}\int_m^\infty dE_{\vec{q}}\,|\vec{q}|\cdot 2\pi i\,\delta\big(p^0 - 2E_{\vec{q}}\big)
\end{aligned}$$

Recall that $\operatorname{Disc}\mathcal{M} = \operatorname{Disc}\delta\mathcal{M}$, since the leading order amplitude has no loop and thus no imaginary part and thus no discontinuity.

Again, we get the very same result, if we replace the propagator in the original amplitude by a δ-function:

$$\frac{1}{(p/2+q)^2 - m^2} \to -2\pi i\,\delta((p/2+q)^2 - m^2).$$

If we go back to the last calculation in (>11.3.3) and replace also the second propagator with a δ-function, we immediately see that the denominator of the propagator will end up as an argument of the δ-function and everything else stays the same:

$$\begin{aligned}
i\,\delta\mathcal{M} &\to \frac{\lambda^2}{2}\int d^4\bar{q}\,\big(-2\pi i\,\delta((p/2+q)^2 - m^2)\big)\big(-2\pi i\,\delta((p/2+q)^2 - m^2)\big)\\
&= -2\pi i\,(-i)\frac{\lambda^2}{4}\frac{4\pi}{(2\pi)^3}\int_m^\infty dE_{\vec{q}}\,|\vec{q}|\,\delta\left(p^0\big(p^0 - 2E_{\vec{q}}\big)\right)\\
&= -2\pi\frac{\lambda^2}{4}\frac{4\pi}{(2\pi)^3}\int_m^\infty dE_{\vec{q}}\,|\vec{q}|\frac{1}{p^0}\,\delta\big(p^0 - 2E_{\vec{q}}\big)\\
&= -\mathrm{Im}\,2\pi\frac{\lambda^2}{4}\frac{4\pi}{(2\pi)^3}\int_m^\infty dE_{\vec{q}}\,|\vec{q}|\frac{1}{p^0}\cdot 2\pi i\,\delta\big(p^0 - 2E_{\vec{q}}\big) = i\operatorname{Disc}\mathcal{M}(p^0).
\end{aligned}$$

Hence, we have proven that we get the discontinuity of the amplitude by replacing the propagator with δ-functions:

$$i\operatorname{Disc}\mathcal{M}(p^0) = \frac{\lambda^2}{2}\int d^4\bar{q}\,\big(-2\pi i\,\delta((p/2+q)^2 - m^2)\big)\big(-2\pi i\,\delta((p/2+q)^2 - m^2)\big).$$

11.3.5 Amplitude with Independent Momenta

The amplitude with k_1, k_2 together with a δ-functin $\delta(k_1 + k_2 - p)$ instead of q is equivalent to the one with q from (>11.3.1):

$$\begin{aligned}
&\frac{\lambda^2}{2}\int d^4\bar{k}_1\,d^4\bar{k}_2\,\frac{1}{k_1^2 - m^2 + i\epsilon}\,\frac{1}{k_2^2 - m^2 + i\epsilon}\,(2\pi)^4\delta(k_1 + k_2 - p)\\
&\quad= \frac{\lambda^2}{2}\int d^4\bar{k}_1\,\frac{1}{k_1^2 - m^2 + i\epsilon}\,\frac{1}{(p - k_1) - m^2 + i\epsilon}\\
&\quad= \frac{\lambda^2}{2}\int d^4\bar{q}\,\frac{1}{(p/2+q)^2 - m^2 + i\epsilon}\,\frac{1}{(p/2-q) - m^2 + i\epsilon} = i\,\delta\mathcal{M},
\end{aligned}$$

where we substituted $k_1 = q + p/2$ in the last step.

11.3.6 Discontinuity with Independent Momenta

If we replace the propagators by δ-function we will get the discontinuity

$$i\operatorname{Disc}\mathcal{M} = \frac{\lambda^2}{2}\int d^4\bar{k}_1\,d^4\bar{k}_2\,\big(-2\pi i\,\delta(k_1^2 - m^2)\big)\,\big(-2\pi i\,\delta(k_2^2 - m^2)\big)\,(2\pi)^4\delta(k_1 + k_2 - p).$$

Now recall from section 4.2 the definition of the Lorentz invariant phase space measure

$$d\tilde{p} \coloneqq d^4\bar{p}\cdot 2\pi\delta(p^2 - m^2)\theta(p^0)$$

and plug it in:[1]

$$i\,\mathrm{Disc}\,\mathcal{M} = -\frac{\lambda^2}{2}\int d\tilde{k}_1\, d\tilde{k}_2\, (2\pi)^4\delta(k_1 + k_2 - p).$$

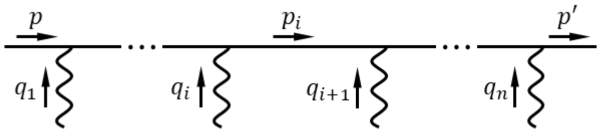
Using $\mathrm{Disc}\,\mathcal{M} = 2i\,\mathrm{Im}\,\mathcal{M}$ from section 11.2, we end up with

$$2\,\mathrm{Im}\,\mathcal{M} = \frac{\lambda^2}{2}\int d\tilde{k}_1\, d\tilde{k}_2\, (2\pi)^4\delta(k_1 + k_2 - p).$$

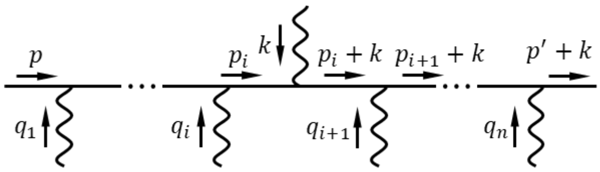
11.5 The Ward-Takahashi Identity

11.5.1 Attachment to an Electron Line between external Electrons

Let's consider an electron line connecting two external electrons with momenta p and p' of one of the diagrams of $\mathcal{M}_0$. An arbitrary number n of internal or external photons with momenta q_i (but not the one with momentum k) may be attached to it:



The momentum of the i-th intermediate electron propagator is then given by $p_i = p + \sum_{j=1}^{i} q_j$. Now suppose we insert the external photon $\gamma(k)$ after the i-th vertex:



The i-th electron propagator p_i is split into two propagators p_i and $p_i + k$ and all other propagators to the right have their momentum increased by k. The contribution of the new vertex is given by $ig\gamma^\mu\varepsilon_\mu = ig\not\varepsilon$. If we replace $\varepsilon \to k$, this turns into $ig\not k$. Including the two adjacent electron propagators, we obtain the expression

$$\begin{aligned}
\frac{i}{\not p_i + \not k - m} ig\not k \frac{i}{\not p_i - m} &= ig\frac{i}{\not p_i + \not k - m}\underbrace{\left((\not p_i + \not k - m) - (\not p_i - m)\right)}_{=\not k}\frac{i}{\not p_i - m} \\
&= ig\left(\frac{i}{\not p_i + \not k - m}(\not p_i + \not k - m)\frac{i}{\not p_i - m} - \frac{i}{\not p_i + \not k - m}(\not p_i - m)\frac{i}{\not p_i - m}\right) \\
&= -g\left(\frac{i}{\not p_i - m} - \frac{i}{\not p_i + \not k - m}\right).
\end{aligned}$$

Including also the next-adjacent vertices and propagators, the structure of the diagram is (dropping the $-g$)

$$\cdots\ \frac{i}{\not p_{i+1} + \not k - m}\ \gamma^{\mu_{i+1}}\ \left(\frac{i}{\not p_i - m} - \frac{i}{\not p_i + \not k - m}\right)\ \gamma^{\mu_i}\ \frac{i}{\not p_{i-1} - m}\ \cdots.$$

If we now attach the photon $\gamma(k)$ not to the i-th but to the $i+1$-th position, the structure of the terms above will change to

$$\cdots\ \left(\frac{i}{\not p_{i+1} - m} - \frac{i}{\not p_{i+1} + \not k - m}\right)\ \gamma^{\mu_{i+1}}\ \frac{i}{\not p_i - m}\ \gamma^{\mu_i}\ \frac{i}{\not p_{i-1} - m}\ \cdots.$$

[1] Somehow, we lack the θ-function in our expression for $\mathrm{Disc}\,\delta\mathcal{M}$. I assume that those are implicitly contained in $\delta(p_1 + p_2 - p)$: Since $p^0 > 0$, p_1^0, p_2^0 should be positive as well.

Note that the second term of this expression cancels the first term of the previous expression. The same cancellation occurs between any other pair of diagrams with adjacent insertions. Only the two insertions at the very beginning and the very end have only one adjacent partner.

At the very beginning (left), an insertion on the left-hand side and right-hand side of q_1 yields

$$\text{left of } q_1: \qquad \frac{i}{\not{p}' + \not{k} - m} \cdots \frac{i}{\not{p}_1 + \not{k} - m} \gamma^{\mu_1} \left(\frac{i}{\not{p} - m} - \frac{i}{\not{p} + \not{k} - m} \right),$$

$$\text{right of } q_1: \qquad \frac{i}{\not{p}' + \not{k} - m} \cdots \left(\frac{i}{\not{p}_1 - m} - \frac{i}{\not{p}_1 + \not{k} - m} \right) \gamma^{\mu_1} \frac{i}{\not{p} - m}.$$

Thus, the term which is *not* cancelled reads

$$- \frac{i}{\not{p}' + \not{k} - m} \cdots \frac{i}{\not{p} + \not{k} - m}.$$

At the very end (right), an insertion on the right-hand side and left-hand side of q_n yields

$$\text{right of } q_n: \qquad \left(\frac{i}{\not{p}' - m} - \frac{i}{\not{p}' + \not{k} - m} \right) \gamma^{\mu_n} \frac{i}{\not{p}_{n-1} - m} \cdots \frac{i}{\not{p} - m},$$

$$\text{left of } q_n: \qquad \frac{i}{\not{p}' + \not{k} - m} \gamma^{\mu_n} \left(\frac{i}{\not{p}_{n-1} - m} - \frac{i}{\not{p}_{n-1} + \not{k} - m} \right) \cdots \frac{i}{\not{p} - m}.$$

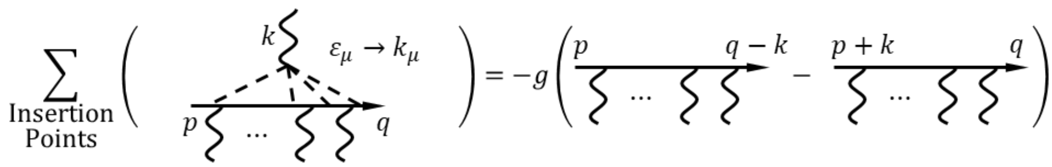
Thus, the term which is *not* cancelled in this case reads

$$\frac{i}{\not{p}' - m} \gamma^{\mu_n} \frac{i}{\not{p}_{n-1} - m} \cdots \frac{i}{\not{p} - m}.$$

If we define $q \coloneqq p' + k$ and add the factor $-g$ we dropped before, the two non-vanishing terms are

$$-g \left(\frac{i}{\not{q} - \not{k} - m} \cdots \frac{i}{\not{p} - m} - \frac{i}{\not{q} - m} \cdots \frac{i}{\not{p} + \not{k} - m} \right).$$

Diagrammatically, we can draw our result as



11.5.2 The Ward Identity

According to the "alternative form of the LSZ reduction formula" of the very end of section 7.5, the contribution of a correlation function/a diagram to a S-matrix element is given by the coefficient of the product of the poles

$$\frac{1}{p_i^2 - m^2}$$

of all external particles (all particles on the mass shell) involved. In our diagrammatic formula, on the left-hand side the poles of the external particles are given by

$$\frac{1}{\not{p} - m} \frac{1}{\not{q} - m} = \frac{\not{p} + m}{p^2 - m^2} \frac{\not{q} + m}{q^2 - m^2}.$$

On the right-hand side, the pole structure is

$$\frac{i}{\not{q} - \not{k} - m} \frac{i}{\not{p} - m} - \frac{i}{\not{q} - m} \frac{i}{\not{p} + \not{k} - m}.$$

Each one of these two terms contain one of the poles of the term on the right-hand side, but neither contains both poles. Thus, the contribution to the S-matrix of the right-hand side vanishes.

12 Regularization

12.2 Feynman Parameters

12.2.1 Feynman Parameters for Two and Three Factors in the Denominator

For $n = 2$ factors in the denominator with $a_1 = a_2 = 1$, we find

$$\prod_{i=1}^{2} \frac{1}{A_i} = \frac{\Gamma\left(\sum_{i=1}^{2} 1\right)}{\prod_{i=1}^{2} \Gamma(1)} \int_0^1 dx_1\, dx_2 \frac{\delta\left(1 - \sum_{i=1}^{2} x_i\right) \prod_{i=1}^{2} x_i^{1-1}}{\left(\sum_{i=1}^{2} A_i x_i\right)^{\sum_{i=1}^{2} 1}} = \underbrace{\frac{\Gamma(2)}{\Gamma^2(1)}}_{=1} \int_0^1 dx_1\, dx_2 \frac{\delta(1 - x_1 - x_2)}{(A_1 x_1 + A_2 x_2)^2}$$
$$= \int_0^1 dx_1 \frac{1}{\left(A_1 x_1 + A_2 (1 - x_1)\right)^2}.$$

To get the formula given in section 12.2, we rename $x_1 \to x$ and $A_1 \to A$ and $A_2 \to B$.

For $n = 3$ factors in the denominator with $a_1 = a_2 = a_3 = 1$, we find

$$\prod_{i=1}^{3} \frac{1}{A_i} = \frac{\Gamma\left(\sum_{i=1}^{3} 1\right)}{\prod_{i=1}^{3} \Gamma(1)} \int_0^1 dx_1\, dx_2\, dx_3 \frac{\delta\left(1 - \sum_{i=1}^{3} x_i\right) \prod_{i=1}^{3} x_i^{1-1}}{\left(\sum_{i=1}^{3} A_i x_i\right)^{\sum_{i=1}^{3} 1}}$$
$$= \underbrace{\frac{\Gamma(3)}{\Gamma^2(1)}}_{=2} \int_0^1 dx_1\, dx_2\, dx_3 \frac{\delta(1 - x_1 - x_2 - x_3)}{(A_1 x_1 + A_2 x_2 + A_3 x_3)^3}.$$

To get the formula given in section 11.2, we rename $(x_1, x_2, x_3) \to (x, y, z)$ and $(A_1, A_2, A_3) \to (A, B, C)$.

12.3 Dirac Algebra

12.3.1 Contractions of γ-Matrices

γ-matrices are defined – also in d dimensions – by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

Contracting with $\eta_{\mu\nu}$ on both sides, we find

$$2d = 2\eta_{\mu\nu}\eta^{\mu\nu} \overset{!}{=} \eta_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = \{\gamma^\mu, \gamma_\mu\} = 2\gamma^\mu\gamma_\mu \qquad \Leftrightarrow \qquad \gamma^\mu\gamma_\mu = d = 4 - \epsilon.$$

Furthermore, we find, using $d = 4 - \epsilon$,

$$\gamma^\mu\gamma^\nu\gamma_\mu = \gamma^\mu\left(\{\gamma^\nu, \gamma_\mu\} - \gamma_\mu\gamma^\nu\right) = \gamma^\mu \cdot 2\eta_\mu^\nu - d\gamma^\nu = (2 - d)\gamma^\nu = (\epsilon - 2)\gamma^\nu,$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = \gamma^\mu\gamma^\nu\left(\{\gamma^\rho, \gamma_\mu\} - \gamma_\mu\gamma^\rho\right) = 2\gamma^\rho\gamma^\nu - (2 - d)\gamma^\nu\gamma^\rho = 2\{\gamma^\rho, \gamma^\nu\} - (4 - d)\gamma^\nu\gamma^\rho$$
$$= 4\eta^{\rho\nu} - (4 - d)\gamma^\nu\gamma^\rho = 4\eta^{\rho\nu} - \epsilon\gamma^\nu\gamma^\rho,$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = \gamma^\mu\gamma^\nu\gamma^\rho\left(\{\gamma^\sigma, \gamma_\mu\} - \gamma_\mu\gamma^\sigma\right) = 2\gamma^\sigma\gamma^\nu\gamma^\rho - (4\eta^{\rho\nu} - (4 - d)\gamma^\nu\gamma^\rho)\gamma^\sigma$$
$$= 2\gamma^\sigma\gamma^\nu\gamma^\rho - 4\gamma^\sigma\left(\frac{1}{2}\{\gamma^\rho, \gamma^\nu\}\right) + (4 - d)\gamma^\nu\gamma^\rho\gamma^\sigma = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4 - d)\gamma^\nu\gamma^\rho\gamma^\sigma$$
$$= -2\gamma^\sigma\gamma^\rho\gamma^\nu + \epsilon\gamma^\nu\gamma^\rho\gamma^\sigma.$$

12.3.2 Traces of Odd Number of γ-Matrices Vanishes

Introducing $\gamma^5 \coloneqq i\gamma^0\gamma^1\gamma^2\gamma^3$, we find, using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$,

$$\gamma^5\gamma^\mu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu = (-1)^3 i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^\mu\gamma^5,$$

$$(\gamma^5)^2 = -(\gamma^0\gamma^1\gamma^2\gamma^3)(\gamma^0\gamma^1\gamma^2\gamma^3) = -\underbrace{\gamma^0\gamma^0}_{=1}\underbrace{\gamma^1\gamma^1}_{=-1}\underbrace{\gamma^2\gamma^2}_{=-1}\underbrace{\gamma^3\gamma^3}_{=-1} = 1.$$

Thus, for an *odd* number of γ-matrices in the trace we find

$$\operatorname{Tr}\gamma^\mu\cdots\gamma^\nu = \operatorname{Tr}\gamma^\mu\cdots\gamma^\nu\underbrace{\gamma^5\gamma^5}_{=1} = -\operatorname{Tr}\gamma^5\gamma^\mu\cdots\gamma^\nu\gamma^5 = -\operatorname{Tr}\gamma^\mu\cdots\gamma^\nu.$$

During the first equal sign, we plugged in a 1 in terms of $(\gamma^5)^2$. During the second equal sign, we commuted the first γ^5-matrix through the odd number of other γ-matrices to the left. For each commutation, we get a factor -1 and since there is an odd number of them, one factor -1 remains. Due to the cyclicity of the trace, we can then reunite the two γ^5 matrices (and let them disappear by $(\gamma^5)^2 = 1$) without receiving any factor of -1. Thus, the trace equals minus itself and thereby vanishes.

12.3.3 Traces of Even Number of γ-Matrices

For traces over an even number of d-dimensional γ-matrices, consider

$$\operatorname{Tr}\gamma^\mu\gamma^\nu = \frac{1}{2}\operatorname{Tr}\{\gamma^\mu,\gamma^\nu\} = \eta^{\mu\nu}\operatorname{Tr}\mathbb{I}_d,$$

$$\begin{aligned}\operatorname{Tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= \operatorname{Tr}\big((2\eta^{\mu\nu}-\gamma^\nu\gamma^\mu)\gamma^\rho\gamma^\sigma\big) = 2\eta^{\mu\nu}\eta^{\rho\sigma}\operatorname{Tr}\mathbb{I}_d - \operatorname{Tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\\ &= 2\eta^{\mu\nu}\eta^{\rho\sigma}\operatorname{Tr}\mathbb{I}_d - \operatorname{Tr}(\gamma^\nu(2\eta^{\mu\rho}-\gamma^\rho\gamma^\mu)\gamma^\sigma)\\ &= 2\eta^{\mu\nu}\eta^{\rho\sigma}\operatorname{Tr}\mathbb{I}_d - 2\eta^{\mu\rho}\eta^{\nu\sigma}\operatorname{Tr}\mathbb{I}_d + \operatorname{Tr}\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\\ &= 2\eta^{\mu\nu}\eta^{\rho\sigma}\operatorname{Tr}\mathbb{I}_d - 2\eta^{\mu\rho}\eta^{\nu\sigma}\operatorname{Tr}\mathbb{I}_d + \operatorname{Tr}\big(\gamma^\nu\gamma^\rho(2\eta^{\mu\sigma}-\gamma^\sigma\gamma^\mu)\big)\\ &= 2\eta^{\mu\nu}\eta^{\rho\sigma}\operatorname{Tr}\mathbb{I}_d - 2\eta^{\mu\rho}\eta^{\nu\sigma}\operatorname{Tr}\mathbb{I}_d + 2\eta^{\mu\sigma}\eta^{\nu\rho}\operatorname{Tr}\mathbb{I}_d - \underbrace{\operatorname{Tr}\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu}_{=\operatorname{Tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma}\end{aligned}$$

$$\Leftrightarrow \qquad \operatorname{Tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})\operatorname{Tr}\mathbb{I}_d.$$

What is $\mathbb{I}_d$? That is, what is the *matrix dimension* of a γ-matrix (how many rows and columns does it have), if we live in d *Minkowski dimensions*? d Minkowski dimensions mean that the indices μ run over d numbers, so there must be d different γ-matrices all obeying their defining relation $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$. Usually, we want to choose a *representation* for the γ-matrix with minimal matrix dimension. In $d = 4$, we need for different matrices obeying the defining relation and the minimal matrix dimension needed is 4. In $d = 2$ and $d = 3$ the Pauli matrices with Matrix dimension 2 do the job. Thus, for example,

$$\operatorname{Tr}\mathbb{I}_d = \text{number of matrix dimensions} = \begin{cases}2, & \text{for } d = 2,3,\\ 4, & \text{for } d = 4.\end{cases}$$

12.3.4 Integrals of Four Momenta

The d-dimensional integral (that is, $\mu = 1,2,\ldots,d$)

$$\int d^d\bar{l}\,\frac{l^\mu}{D(l^2)}$$

vanishes, since the denominator $D(p^2)$ is an even function of p^μ whereas the numerator is an odd function of p^μ.

The integral

$$\int d^d\bar{l}\,\frac{l^\mu l^\nu}{D(l^2)}$$

vanishes by the same symmetry argument if $\mu \neq \nu$. This already suggests that we can substitute $l^\mu l^\nu \to c l^2\eta^{\mu\nu}$ (we need l^2 to get the dimensions right). After integration over l, something must carry the Lorentz indices μ,ν and what else could it be than $\eta^{\mu\nu}$? There is no other tensor involved. Moreover, this is the only choice to preserve Lorentz invariance. However, there could be in general a prefactor $c \in \mathbb{R}$ when doing this substitution. We can find it, when contracting both sides of the equation with $\eta_{\mu\nu}$:

$$\int d^d\bar{l}\frac{l^\mu l^\nu}{D(l^2)} = c\eta^{\mu\nu}\int d^d\bar{l}\frac{l^2}{D(l^2)} \qquad \overset{\cdot\eta_{\mu\nu}}{\Leftrightarrow} \qquad \int d^d\bar{l}\frac{l^2}{D(l^2)} = c\eta_{\mu\nu}\eta^{\mu\nu}\int d^d\bar{l}\frac{l^2}{D(l^2)}.$$

For consistence, we obviously need

$$\frac{1}{c} = \eta^{\mu\nu}\eta_{\mu\nu} = \eta^\mu_\mu = d.$$

12.3.5 Gordon Identities

We can proof this in the following way: First, we note that, using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$,

$$\sigma^{\mu\nu} \coloneqq \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{2}\big(\gamma^\mu\gamma^\nu - (\{\gamma^\mu,\gamma^\nu\} - \gamma^\mu\gamma^\nu)\big) = i(\gamma^\mu\gamma^\nu - \eta^{\mu\nu}).$$

Plugging this result in, we find

$$\begin{aligned}
&\bar{u}_k((k \mp p)^\mu X + i\sigma^{\mu\nu}(k \pm p)_\nu X)u_p \\
&\qquad = \bar{u}_k((k \mp p)^\mu X - (\gamma^\mu\gamma^\nu - \eta^{\mu\nu})(k \pm p)_\nu X)u_p \\
&\qquad = \bar{u}_k(2k^\mu X - \gamma^\mu\gamma^\nu k_\nu X \mp \gamma^\mu \not{p} X)u_p \\
&\qquad = \bar{u}_k(2k^\mu X - (\{\gamma^\mu,\gamma^\nu\} - \gamma^\nu\gamma^\mu)k_\nu X \mp \gamma^\mu \not{p} X)u_p \\
&\qquad = \bar{u}_k(\not{k}\gamma^\mu X \mp \gamma^\mu \not{p} X)u_p.
\end{aligned}$$

Let us now apply $\not{p}u_p = mu_p$ and $\bar{u}_k\not{k} = \bar{u}_k m$. For that, we need to commute $\not{p}$ through X. Thus, in the case of $X = \gamma^5$, we get an additional minus sign and end up with:

$$\bar{u}_k((k \mp p)^\mu X + i\sigma^{\mu\nu}(k \pm p)_\nu X)u_p = \begin{cases} \bar{u}_k(m\gamma^\mu \mp \gamma^\mu m)u_p, & \text{for } X = \mathbb{I} \\ \bar{u}_k(m\gamma^\mu\gamma^5 \pm \gamma^\mu\gamma^5 m)u_p, & \text{for } X = \gamma^5 \end{cases}.$$

In the case of v spinors, the derivation is exactly the same. The only difference is, that for v spinors $\not{p}v_p = -mv_p$ and $\bar{v}_k\not{k} = -\bar{v}_k m$. Thus, we get an overall minus sign.

12.4 Wick Rotation

12.4.1 Wick Rotation

We stated in section 12.2, that we can shift the integration variable (the loop momentum) $k^\mu \to l^\mu + \cdots$ in such a way, that the numerator of intgrang will become $(l^2 - \Delta + i\epsilon)^a$. By the Dirac algebra of section 12.3 (especially by the integrals of four-momenta in that section), also the numerator will be simplified significantly: The integration variable l^μ will appear as l^2 only (we have proved this for numerator terms $\sim l^\mu$ and $\sim l^\mu l^\nu$ in section 12.3). Our loop integral will by now be of the form

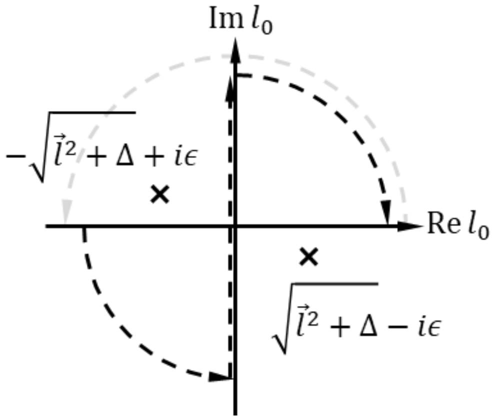
$$\int dx_1 \cdots dx_n \int d^4\bar{l}\, \frac{f(l^2)}{(l^2 - \Delta + i\epsilon)^2}\delta(1 - \Sigma_{i=1}^n x_i),$$

where Δ, f can contain quantities like Feynman parameters or other momenta; that is, quantities that are constants with respect to the $d^4\bar{l}$ integral.

We now want to evaluate the momentum integral over l. The integrand only depends on l^2, but not on individual components l^μ. Thus, if it were not for the minus signs in the Minkowski metric, we could perform the entire four-dimensional integral in four-dimensional spherical coordinates. To remove the minus sigs, consider the following: The integration over the l_0-component has two poles at

$$l_0^2 - \vec{l}^2 - \Delta + i\epsilon = 0 \qquad \Leftrightarrow \qquad l_0 = \pm\sqrt{\vec{l}^2 + \Delta - i\epsilon} = \pm\sqrt{\vec{l}^2 + \Delta} \mp i\tilde{\epsilon} + \mathcal{O}(\tilde{\epsilon}^2),$$

where $\tilde{\epsilon} \coloneqq \epsilon/2\sqrt{\vec{l}^2 + \Delta}$ is still arbitrarily small (we will call $\tilde{\epsilon}$ again ϵ from now on). If the integral falls off sufficiently rapidly at large $|l^0|$ in the complex plane, we can close the contour in the complex plane at infinity without changing the integral since the integral is zero at infinity (light grey arrow in the figure). Then we use the residue theorem due to which the form of the contour is irrelevant as long as the same poles stay inside. Thus, without changing the value of the integral, we can instead of integrating along the real axis integrate along the imaginary axis. To get there, the contour now also includes two infinitely large quarter circles, namely in the first and third quadrant (black dashed arrows). Now, the two quarter circles and the halve circle of this new closed contour do not contribute, since they are at infinity. Thus, the integral along the real axis is just the same as the integral along the imaginary axis.



To achieve this mathematically, we substitute $l^0 = il_E^0$ and $\vec{l} = \vec{l}_E$. If we then integrate l_E^0 from $-\infty$ to ∞, the integral goes along the imaginary axis of l^0, as desired. l_E is now a Euclidean four-dimensional vector in the sense that

$$l^2 = (l^0)^2 - \vec{l}^2 = (il_E^0)^2 - \vec{l}_E^2 = -\big((l_E^0)^2 + \vec{l}_E^2\big) =: -l_E^2, \qquad d^4l = i\, d^4 l_E.$$

Since the new integration contour along the imaginary axis does not pass the poles (in the limit $\epsilon \to 0$), we can perform this limit without effecting the integral.

12.6 Dimensional Regularization

12.6.1 Surface of a d-Dimensional Unit Sphere

To compute the surface of a d-dimensional unit sphere, consider Gaussian integrals:

$$\sqrt{\pi}^d = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^d = \int_{-\infty}^{\infty} dx_1 \cdots dx_d\, e^{-x_1^2} \cdots e^{-x_d^2} = \int d^d x\, e^{-\vec{x}^2} = \int d\Omega_d \int_0^\infty x^{d-1}\, dx\, e^{-x^2},$$

where $\vec{x}$ was the d-dimensional vector $\vec{x} = (x_1, \dots, x_d)^T$. We can now substitute $z = x^2 \Longrightarrow dz = 2x\, dx$ and find

$$\sqrt{\pi}^d = \int d\Omega_d \cdot \frac{1}{2}\int_0^\infty z^{d/2-1}\, dz\, e^{-z} = \int d\Omega_d \cdot \frac{1}{2}\Gamma(d/2) \qquad \Leftrightarrow \qquad \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} =: \Omega_d,$$

where

$$\Gamma(x) = \int_0^\infty dz\, z^{x-1}\, e^{-z}$$

is the gamma function. Using $d = 4 - \epsilon$, we find

$$\Omega_d = \Omega_{4-\epsilon} = \frac{2\pi^{2-\epsilon/2}}{\Gamma(2-\epsilon/2)}.$$

12.6.2 The Radial Integrals in d Dimensions

The formulas can also be given in the form

$$\int \frac{d^d \bar{l}_E}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2}},$$

$$\int \frac{d^d \bar{l}_E\, l_E^2}{(l_E^2+\Delta)^n} = \frac{d/2}{(4\pi)^{d/2}} \frac{\Gamma(n-1-d/2)}{\Gamma(n)} \frac{1}{\Delta^{n-1-d/2}}.$$

Let us now prove the first of the two integrals and only in the special case of $n = 2$.[1] Using that Ω_d is the volume of the unit sphere, we switch to d-dimensional spherical coordinates (the factor $(2\pi)^d$ drops out of $\bar{l}_E$):

$$I := \int d^d \bar{l}_E \frac{1}{(l_E^2+\Delta)^2} = \frac{\Omega_d}{(2\pi)^d} \int_0^\infty l_E^{d-1}\, dl_E\, \frac{1}{(l_E^2+\Delta)^2} = \frac{\Omega_d}{(2\pi)^d} \frac{1}{2} \int_0^\infty x^{d/2-1}\, dx\, \frac{1}{(x+\Delta)^2},$$

where we substituted $x := l_E^2 \Longrightarrow dx = 2 l_E\, dl_E$. Next, we substitute

$$z := \frac{\Delta}{x+\Delta} \qquad \Longrightarrow \qquad dz = \frac{-\Delta}{(x+\Delta)^2} dx, \qquad x = \Delta\left(\frac{1}{z} - 1\right) = \Delta\,(1-z)\,z^{-1}.$$

This yields

$$\begin{aligned} I &= \frac{\Omega_d}{(2\pi)^d} \frac{\Delta^{d/2-1}}{2} \int_1^0 (1-z)^{d/2-1}\, z^{-d/2+1} \left(dz \frac{(x+\Delta)^2}{-\Delta}\right) \frac{1}{(x+\Delta)^2} = \\ &= \frac{\Omega_d}{(2\pi)^d} \frac{\Delta^{d/2-2}}{2} \int_0^1 dz\,(1-z)^{d/2-1}\, z^{-d/2+1}. \end{aligned}$$

We now use the definition of the beta function as well as its mathematical connection to the gamma function:

$$B(\alpha,\beta) = \int_0^1 dx\, x^{\alpha-1}\,(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

In our case, $\alpha = 2 - d/2$ and $\beta = d/2$; hence

$$I = \frac{\Omega_d}{(2\pi)^d} \frac{\Delta^{d/2-2}}{2} \frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} \frac{1}{\Delta^{2-d/2}} = \frac{1}{(4\pi)^{2-\epsilon/2}} \frac{\Gamma(\epsilon/2)}{\Gamma(2)} \frac{1}{\Delta^{\epsilon/2}},$$

where we plugged in the formula for Ω_d from (>12.6.1) as well as $d = 4 - \epsilon$.

12.6.3 Series Expansion of the Gamma Function

Let's start with the simpler task, how to expand a^ϵ:

$$b^{a\epsilon} = e^{a\epsilon \ln b} = 1 + a\epsilon \ln b + \mathcal{O}(\epsilon^2).$$

Alright, and now let's focus on the gamma function: Using $x\Gamma(x) = \Gamma(x+1)$, we can expand the gamma function $\Gamma(a\epsilon)$ at $\epsilon = 0$ in the following way:

$$\Gamma(a\epsilon) = \frac{1}{a\epsilon}\Gamma(a\epsilon+1) = \frac{1}{a\epsilon}\big(\Gamma(1) + \Gamma'(1)a\epsilon + \mathcal{O}(\epsilon^2)\big) = \frac{1}{a\epsilon} - \gamma + \mathcal{O}(\epsilon).$$

[1] According to Pesking & Schroeder, the general formulas can easily be verified. I was not able to do that, however, I did not try too hard.

We used here that $\Gamma(1)=1$ and the derivative of the gamma function $\Gamma(x)$ at $x=1$ is given by -1 times the *Euler-Mascheroni constant* $\gamma \approx 0.577$.

Note that we obviously have

$$\Gamma(m+a\epsilon) \overset{\epsilon\to 0}{=} \Gamma(b) \qquad \text{for} \qquad m>0.$$

Next, consider $\Gamma(-m+a\epsilon)$ for some $m \in \mathbb{N} \setminus 0$. We can pull as many arguments of Γ in front of it using $\Gamma(x) = x^{-1}\Gamma(x+1)$ until the argument becomes positive when $\epsilon \to 0$. In the denominators (that is, the factors x^{-1}), we can directly set $\epsilon \to 0$ – except for the last one:

$$\begin{aligned}
\Gamma(-m+a\epsilon) &= \frac{1}{-m}\cdot\frac{1}{-m+1}\cdots\frac{1}{-m+(m-1)}\cdot\frac{1}{-m+a\epsilon+m}\Gamma(a\epsilon+1) \\
&= \frac{1}{(-m)\cdot\big(-(m-1)\big)\cdots(-1)}\cdot\frac{1}{a\epsilon}\Gamma(a\epsilon+1) = \frac{(-1)^m}{m!\,a\epsilon}\Gamma(a\epsilon+1) \\
&= \frac{(-1)^m}{m!}\left(\frac{1}{a\epsilon}-\gamma+\mathcal{O}(\epsilon)\right),
\end{aligned}$$

where in the last step, we expanded $\Gamma(a\epsilon+1)$ as above. Comparing to the expansion of $\Gamma(a\epsilon)$, this formula obviously contains also the case $m=0$. For us, only the special case $a=1/2$ will be relevant.

12.6.4 Expanding the Integral

We want to compute this integral in d dimensions, that is we convert

$$g^2\int d^4\bar{l}_E\,\frac{1}{(l_E^2+\Delta)^2} \to g^2\mu^\epsilon\int d^d\bar{l}_E\,\frac{1}{(l_E^2+\Delta)^2}.$$

Using the formulas that we derived so far in section 12.6, we find

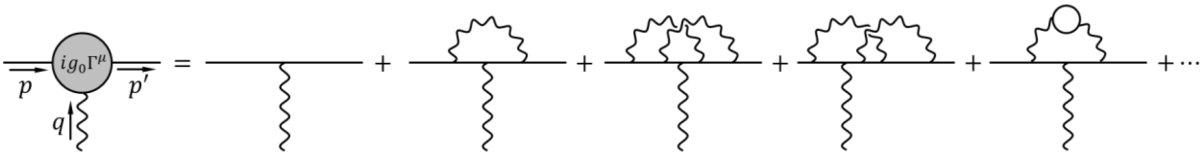
$$\begin{aligned}
g^2\mu^\epsilon\int d^d\bar{l}_E\,\frac{1}{(l_E^2+\Delta)^2} &= \frac{g^2\mu^\epsilon}{(4\pi)^{2-\frac{\epsilon}{2}}}\frac{\Gamma(\epsilon/2)}{\Gamma(2)}\frac{1}{\Delta^{\epsilon/2}} \\
&= \frac{g^2}{(4\pi)^2}\Gamma(\epsilon/2)\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon/2} \\
&= \frac{g^2}{(4\pi)^2}\left(\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)\right)\left(1+\frac{\epsilon}{2}\ln\frac{4\pi\mu^2}{\Delta}+\mathcal{O}(\epsilon^2)\right) \\
&= \frac{g^2}{(4\pi)^2}\left(\frac{2}{\epsilon}+\ln\frac{4\pi\mu^2}{\Delta}-\gamma+\mathcal{O}(\epsilon)\right) \\
&= \frac{g^2}{(4\pi)^2}\left(\frac{2}{\epsilon}+\ln\frac{4\pi\mu^2e^{-\gamma}}{\Delta}+\mathcal{O}(\epsilon)\right) \\
&= \frac{g^2}{(4\pi)^2}\left(\frac{2}{\epsilon}+\ln\frac{\tilde{\mu}^2}{\Delta}+\mathcal{O}(\epsilon)\right).
\end{aligned}$$

13 DIVERGENCES IN QED

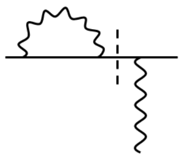
13.2 The Vertex Correction

13.2.1 Diagrams Contributing to the Vertex Correction

The blob Γ^μ is the sum of all diagrams that cannot be split into two diagrams by removing a single line. That is, it contains, for example, the following term:



However, it does not contain a term like



since cutting along the dashed line would split the diagram into two separate diagrams. This is part of what we mean by $ig_0\Gamma^\mu$ an "amputated" amplitude. What we also mean by that, is that $ig_0\Gamma^\mu$ does not contain propagators, spinors or polarizations vectors of its external lines.

13.2.2 General Form of the Vertex Amplitude

The structure of the object Γ^μ is far from arbitrary. The Feynman rules only allow for the appearance of $\gamma^\mu, p^\mu, p'^\mu$ and constants like m or g_0 within Γ^μ. Of course, *each term* within Γ^μ must carry exactly *one* index, namely μ. Thus, the most general form is

$$\Gamma^\mu = A\gamma^\mu + \tilde{B}p'^\mu + \tilde{C}p^\mu = A\gamma^\mu + B(p'^\mu + p^\mu) + C(p'^\mu - p^\mu).$$

It will turn out to be convenient to work with the combinations $p'^\mu \pm p^\mu$ instead of p^μ and p'^μ.

Aside from constant numbers, the coefficients A, B, C can involve scalar products of γ^μ, p^μ and p'^μ. However, scalar products between two γ-matrices can always be reduced to numbers, like in the simplest case of $\gamma_\mu\gamma^\mu = 4$ (see section 12.3). Scalar products between a γ-matrix and a momentum, that is $\not{p}$ and $\not{p}'$ reduce to m when applied to an adjacent spinor: $\not{p}u_p = mu_p$ and $\bar{u}_{p'}\not{p}' = m\bar{u}_{p'}$. Thus, without loss of generality, we can assume that A, B, C do not contain any γ-matrices. Trivially, we can also ignore the scalar products of momenta with themselves, since they just give the masses $p^2 = p'^2 = m^2$. The only thing left is $p' \cdot p$ or, equivalently, $q^2 = (p' - p)^2 = -2p' \cdot p + 2m^2$. Hence, the coefficients A, B, C must be functions of q^2 (and of constants like m) only.

The Ward identity from section 11.5 will help us to further simplify the structure of Γ^μ. It implies

$$\bar{u}_{p'}\Gamma^\mu u_p\, q_\mu \stackrel{!}{=} 0 \qquad \Leftrightarrow \qquad q_\mu\Gamma^\mu \stackrel{!}{=} 0.$$

Let's see what happens if we apply q_μ to the three terms of our Γ^μ:

$$q_\mu\, A\gamma^\mu = A\not{q} = A(\not{p}' - \not{p}) \to 0,$$

$$\begin{aligned} q_\mu\, B(p'^\mu + p^\mu) &= B(q \cdot p' + q \cdot p) = B\big((p' - p) \cdot p' + (p' - p) \cdot p\big) \\ &= B(m^2 - p \cdot p' + p' \cdot p - m^2) = 0, \end{aligned}$$

$$\begin{aligned} q_\mu\, C(p'^\mu - p^\mu) &= C(q \cdot p' - q \cdot p) = C\big((p' - p) \cdot p' - (p' - p) \cdot p\big) \\ &= C(m^2 - p \cdot p' - p' \cdot p + m^2) = C(2m^2 - 2p' \cdot p) = C(p' - p)^2 = Cq^2. \end{aligned}$$

The first term will vanish when sandwiched between $\bar{u}_{p'}$ and u_p due to $\not{p}u_p = mu_p$ and $\bar{u}_{p'}\not{p}' = m\bar{u}_{p'}$. The second term vanishes directly, only the last is left over. Since the sum of all three terms *must* vanish due to the Ward identity, we find $C = 0$. Thus, we have arrived at the form

$$\Gamma^\mu = A\gamma^\mu + B(p'^\mu + p^\mu).$$

No further true simplifications are possible. However, it is conventional to bring this expression into a different shape, using the Gordon identity from section 12.3. Taking its version with $X = \mathbb{I}$ and the lower sign of $\pm/\mp$, it reads (for p' instead of k)

$$\bar{u}_{p'}((p'+p)^\mu + i\sigma^{\mu\nu}(p'-p)_\nu)u_p = m\,\bar{u}_{p'}(\gamma^\mu + \gamma^\mu)u_p$$

$$\Leftrightarrow \qquad \bar{u}_{p'}\gamma^\mu u_p = \bar{u}_{p'}\left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u_p.$$

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. Thereby, we can use it to swap the $p'^\mu + p^\mu$-term with a $\sigma^{\mu\nu}q_\nu$ term (since Γ^μ is sandwiches by u_p spinors):

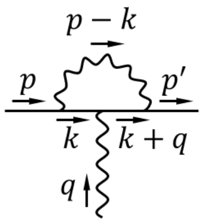
$$\Gamma^\mu = A\gamma^\mu + B(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2),$$

where

$$F_1(q^2) = A + 2mB, \qquad F_2(q^2) = -2mB.$$

13.2.3 Amplitude of the NLO Vertex Correction by Feynman Rules

Using Feynman rules, we want to note down the vertex in the diagram



Using $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu + \cdots$, we want to evaluate the quantity $\bar{u}_{p'}\delta\Gamma^\mu u_p$, as it appears in the amplitude $\mathcal{M}$ of the complete Feynman diagram from the beginning of section 13.2. Note that one factor ig was already denoted outside of $\delta\Gamma^\mu$, which is why the three vertices of the diagram above give us only two factors of ig. Thus, we find[1,2]

$$\begin{aligned}\bar{u}_{p'}\delta\Gamma^\mu u_p &= \int d^4\bar{k}\;\; \bar{u}_{p'}\;\; ig\gamma^\nu \frac{-i\eta_{\nu\sigma}}{(p-k)^2 - \nu^2 - \Lambda^2 + i\epsilon}\; \frac{i}{\not{k} + \not{q} - m + i\epsilon}\gamma^\mu \frac{i}{\not{k} - m + i\epsilon} ig\gamma^\sigma \;\; u_p \\ &= -ig^2 \int d^4\bar{k} \frac{\bar{u}_{p'}\;\; \gamma^\nu(\not{k} + \not{q} + m)\gamma^\mu(\not{k} + m)\gamma_\nu \;\; u_p}{((p-k)^2 - \nu^2 - \Lambda^2 + i\epsilon)((k+q)^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} \\ &= -ig^2 \int d^4\bar{k} \frac{\bar{u}_{p'}\;\; (\gamma^\nu(\not{k} + \not{q})\gamma^\mu\not{k}\gamma_\nu + m\gamma^\nu(\not{k} + \not{q})\gamma^\mu\gamma_\nu + m\gamma^\nu\gamma^\mu\not{k}\gamma_\nu + m^2\gamma^\nu\gamma^\mu\gamma_\nu)\;\; u_p}{((p-k)^2 - \nu^2 - \Lambda^2 + i\epsilon)((k+q)^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)}.\end{aligned}$$

---

[1] Using our standard trick

$$\frac{1}{\not{p} - m + i\epsilon} = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}.$$

[2] Note, that we already added here a Pauli-Villars regulator $-\Lambda^2$. In (>13.2.7) we will see, that the momentum loop integral is ultraviolet divergent and we therefore need to regularize it; we will choose Pauli-Villars regularization for that. Thus, we are going to need the expression for the amplitude including this regulator.
Similarly, since this integral is also infrared divergent, we added a small photon mass $-\nu^2$ to regulate an infrared divergence.

Before we are going to introduce Feynman parameters in (>13.2.4), let us already simplify this expression a little bit with some Dirac algebra from section 12.3, namely $\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu$ and $\gamma^\nu\gamma^\sigma\gamma^\mu\gamma_\nu = 4\eta^{\mu\sigma}$ and $\gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho\gamma_\nu = -2\gamma^\rho\gamma^\mu\gamma^\sigma$. Thereby, we find

$$\begin{aligned}\bar{u}_{p'}\delta\Gamma^\mu u_p &= -ig^2 \int d^4\bar{k} \frac{\bar{u}_{p'}\ \left(-2\not{k}\gamma^\mu(\not{k}+\not{q}) + 4m(k+q)^\mu + 4mk^\mu - 2m^2\gamma^\mu\right)\ u_p}{((p-k)^2 - \nu^2 - \Lambda^2 + i\epsilon)((k+q)^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} \\ &= 2ig^2 \int d^4\bar{k} \frac{\bar{u}_{p'}\ \left(\not{k}\gamma^\mu(\not{k}+\not{q}) - 2m(2k+q)^\mu + m^2\gamma^\mu\right)\ u_p}{((p-k)^2 - \nu^2 - \Lambda^2 + i\epsilon)((k+q)^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)}.\end{aligned}$$

13.2.4 Introducing Feynman parameters

In section 12.2, we found the following formula for introducing Feynman parameters for a denominator of three factors:

$$\frac{1}{ABC} = 2\int_0^1 dx\, dy\, dz \frac{\delta(1-x-y-z)}{(Ax+By+Cz)^3}.$$

In our case, this formula transforms our denominator in the following way:

$$\begin{aligned}&\frac{1}{((p-k)^2 - \nu^2 - \Lambda^2 + i\epsilon)((k+q)^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} \\ &\quad = 2\int_0^1 dx\, dy\, dz \frac{\delta(1-x-y-z)}{D^3},\end{aligned}$$

where D is shorthand for

$$\begin{aligned}D &:= ((p-k)^2 - \nu^2 - \Lambda^2 + i\epsilon)x + ((k+q)^2 - m^2 + i\epsilon)y + (k^2 - m^2 + i\epsilon)z \\ &= k^2 + 2k\cdot(qy - px) + p^2x + q^2y - m^2(y+z) - x\nu^2 - x\Lambda^2 + i\epsilon.\end{aligned}$$

Here, we made use of the δ-function in the integral, due to which $x+y+z=1$. If we now define

$$l := k + qy - px \qquad \Leftrightarrow \qquad k = l - (qy - px),$$

our D depends on l^2 only (and not on other powers of l):

$$\begin{aligned}D &= l^2 - 2l\cdot(qy-px) + (qy-px)^2 + 2l\cdot(qy-px) - 2(qy-px)^2 + p^2x + q^2y \\ &\qquad\qquad - m^2(y+z) - x\nu^2 - x\Lambda^2 + i\epsilon \\ &= l^2 - (qy-px)^2 + p^2x + q^2y - m^2(y+z) - x\nu^2 - x\Lambda^2 + i\epsilon \\ &= l^2 - q^2(y^2-y) + 2p\cdot q\, xy - m^2(x^2 - x + y + z) - x\nu^2 - x\Lambda^2 + i\epsilon \\ &= l^2 - q^2y(-x-z) + 2p\cdot q\, xy - m^2(x^2 - 2x + 1) - x\nu^2 - x\Lambda^2 + i\epsilon \\ &= l^2 + q^2yz + \underbrace{q^2xy + 2p\cdot q\, xy}_{=0} - m^2(1-x)^2 + i\epsilon \\ &= l^2 + q^2yz - m^2(1-x)^2 - x\nu^2 - x\Lambda^2 + i\epsilon,\end{aligned}$$

where in the last step it was used that

$$q^2 + 2p\cdot q = q^2 + 2p\cdot q = q^2 + 2p\cdot q + m^2 - m^2 = (q+p)^2 - m^2 = p'^2 - m^2 = 0.$$

If we assign the abbreviation N to the numerator of the expression for $\bar{u}_{p'}\delta\Gamma^\mu u_p$, we now found[1]

$$\bar{u}_{p'}\delta\Gamma^\mu u_p = 2ig^2\int d^4\bar{l}\int_0^1 dx\, dy\, dz\, 2\delta(1-x-y-z)\frac{N}{D^3},$$

$$\begin{aligned}\text{where}\qquad &D = l^2 - \Delta_\Lambda + i\epsilon, \qquad \Delta_\Lambda := -q^2yz + m^2(1-x)^2 + x\nu^2 + x\Lambda^2, \\ &N := \bar{u}_{p'}\ \left(\not{k}\gamma^\mu(\not{k}+\not{q}) - 2m(2k+q)^\mu + m^2\gamma^\mu\right)\ u_p.\end{aligned}$$

[1] Obviously, $d^4\bar{k} = d^4\bar{l}$. Note that N still contains k, which is to be view as a function of l in this context: $k(l)$. In the next step, we will change from k to l in the numerator as well.

13.2.5 Simplifying the Numerator with Dirac Algebra

In (>13.2.4), we introduced the substitution $k = l - (qy - px)$ for the denominator. Of course, we also need to shift the numerator it that way at the same time. After doing that, recall from section 12.3, that

$$\int d^4\bar{l}\frac{l^\mu}{D^3} = 0, \qquad \int d^4\bar{l}\frac{l^\mu l^\nu}{D^3} = \frac{1}{4}\eta^{\mu\nu}\int d^d\bar{l}\frac{l^2}{D^3}.$$

Hence, at the first arrow "→", we simply plug in $k = l - qy + px$. At the second arrow "→", we use our integration considerations to replace $l^\mu \to 0$ and $l^\mu l^\nu \to \eta^{\mu\nu}l^2/4$:

$$\begin{aligned} N &:= \bar{u}_{p'}(\not{k}\gamma^\mu(\not{k}+\not{q}) - 2m(2k+q)^\mu + m^2\gamma^\mu)u_p \\ &\to \bar{u}_{p'}\big((\not{l}-\not{q}y+\not{p}x)\gamma^\mu(\not{l}-\not{q}y+\not{p}x+\not{q}) - 2m(2l-2qy+2px+q)^\mu + m^2\gamma^\mu\big)u_p \\ &= \bar{u}_{p'}\big(\not{l}\gamma^\mu\not{l} + \not{l}\gamma^\mu(-\not{q}y+\not{p}x+\not{q}) + (-\not{q}y+\not{p}x)\gamma^\mu\not{l} + (-\not{q}y+\not{p}x)\gamma^\mu(-\not{q}y+\not{p}x+\not{q}) \\ &\qquad\quad - 2m(-2qy+2px+q)^\mu + m^2\gamma^\mu\big)u_p \\ &\to \bar{u}_{p'}\Bigg(\frac{1}{4}l^2\underbrace{\eta_{\nu\sigma}\gamma^\nu\gamma^\mu\gamma^\sigma}_{=-2\gamma^\mu} + (-\not{q}y+\not{p}x)\gamma^\mu\big((1-y)\not{q}+\not{p}x\big) - 2m\big((1-2y)q^\mu + 2p^\mu x\big) \\ &\qquad\quad + m^2\gamma^\mu\Bigg)u_p \\ &= \bar{u}_{p'}\Bigg(-\frac{1}{2}l^2\gamma^\mu + \underbrace{(-\not{q}y+\not{p}x)\gamma^\mu\big((1-y)\not{q}+\not{p}x\big)}_{=:\mathcal{A}} - 2m\big((1-2y)q^\mu + 2p^\mu x\big) + m^2\gamma^\mu\Bigg)u_p \end{aligned}$$

From (>13.2.2), we know that the general form of Γ^μ and thus also $\delta\Gamma^\mu$ must be[1]

$$\Gamma^\mu = A\gamma^\mu + B(p'^\mu + p^\mu) + C\underbrace{q^\mu}_{=p'^\mu - p^\mu}.$$

Obviously, the $-l^2/2$- as well as the m^2-term will contribute to A. Let us then take a closer look at the term $\mathcal{A}$. Always remembering that those terms are sandwiched between the u-spinors,[2] we use

$$\not{p}u_p = mu_p, \qquad \bar{u}_{p'}\not{p}' = \bar{u}_{p'}m \qquad \Longrightarrow \qquad \bar{u}_{p'}\not{q}u_p = 0,$$

$$\not{a}\gamma^\mu = -\gamma^\mu\not{a} + 2a^\mu, \qquad \gamma^\mu\not{a} = -\not{a}\gamma^\mu + 2a^\mu \qquad \Longrightarrow \qquad \not{a}\not{b} = -\not{b}\not{a} + 2a\cdot b$$

where the last to identities were derived in (>10.4.1). Then,

$$\mathcal{A} = (-\not{q}y+\not{p}x)\gamma^\mu\big((1-y)\not{q}+\not{p}x\big) = \underbrace{-y(1-y)\not{q}\gamma^\mu\not{q}}_{=:\mathcal{A}_1}\ \underbrace{-\,xy\not{q}\gamma^\mu\not{p}}_{=:\mathcal{A}_2} + \underbrace{x(1-y)\not{p}\gamma^\mu\not{q}}_{=:\mathcal{A}_3} + \underbrace{x^2\not{p}\gamma^\mu\not{p}}_{=:\mathcal{A}_3},$$

$$\mathcal{A}_1 = -y(1-y)(-\gamma^\mu\not{q} + 2q^\mu)\not{q} \to y(1-y)\gamma^\mu q^2,$$

$$\mathcal{A}_2 \to -xy(-\underbrace{\gamma^\mu\not{q}}_{=\gamma^\mu\not{p}'-\gamma^\mu\not{p}=-\not{p}'\gamma^\mu+2p'^\mu-\gamma^\mu\not{p}\to-2m\gamma^\mu+2p'^\mu} + 2q^\mu)m = -2xym(m\gamma^\mu - p'^\mu + q^\mu),$$

$$\begin{aligned} \mathcal{A}_3 &= x(1-y)(-\gamma^\mu\not{p} + 2p^\mu)\not{q} \to -x(1-y)\gamma^\mu\not{p}\not{q} = -x(1-y)\gamma^\mu(-\not{q}\not{p} + 2p\cdot q) \\ &\to -x(1-y)\,(-m\underbrace{\gamma^\mu\not{q}}_{\substack{\to-2m\gamma^\mu+2p'^\mu \\ \text{(siehe }\mathcal{A}_2)}} + \gamma^\mu\underbrace{2p\cdot q}_{=2p\cdot(p'-p)=2(p\cdot p'-m^2)=-(p'-p)^2=-q^2}) = -x(1-y)(2m^2\gamma^\mu - 2mp'^\mu - q^2\gamma^\mu), \end{aligned}$$

$$\mathcal{A}_4 = x^2(-\gamma^\mu\not{p} + 2p^\mu)\not{p} \to x^2(-\gamma^\mu m + 2p^\mu)m$$

---

[1] In (>13.2.2) we started from this expression in terms of A, B, C and then went on and derived an expression in terms of the formfactors $F_1(q^2)$ and $F_2(q^2)$ instead. In the end, we will use the formfactors here as well, but for now it is easier to compute $\delta\Gamma^\mu$ in terms of A, B, C. The transition from A, B, C to $F_1(q^2), F_2(q^2)$ will be very easy.

[2] When we use the fact that the expressions are sandwiched between u-spinors, we are going to use a little arrow "→" instead of an equal sign "=".

$$
\begin{aligned}
\Longrightarrow \qquad \mathcal{A} &= \gamma^\mu \cdot (y(1-y)q^2 - 2xym^2 - x(1-y)(2m^2-q^2) - x^2m^2) \\
&\quad + q^\mu \cdot \ (-2xym) \\
&\quad + p'^\mu \cdot (2xym + 2x(1-y)m) \\
&\quad + p^\mu \cdot (2x^2m) \\
&= \gamma^\mu \cdot \big((x+y)(1-y)q^2 - xm^2(2+x)\big) \\
&\quad + q^\mu \cdot \ (-2xym) \\
&\quad + p'^\mu \cdot (2xm) \\
&\quad + p^\mu \cdot (2x^2m)
\end{aligned}
$$

Using $x+y+z=1$, we arrive at, the following $\widetilde{N}$ with $N = \bar{u}_{p'}\widetilde{N}u_p$:

$$
\begin{aligned}
\widetilde{N} &= \gamma^\mu \cdot \left(-\frac{1}{2}l^2 + (x+y)(1-y)q^2 - xm^2(2+x) + m^2\right) \\
&\quad + q^\mu \cdot \big(-2xym - 2m(1-2y)\big) \\
&\quad + p'^\mu \cdot 2xm \\
&\quad + p^\mu \cdot (2x^2m - 4mx) \\
&= \gamma^\mu \cdot \left(-\frac{1}{2}l^2 + (1-z)(1-y)q^2 + (1-2x-x^2)m^2\right) \\
&\quad + q^\mu \cdot \big(-2xym - 2m(1-2y)\big) \\
&\quad + p'^\mu \cdot 2xm \\
&\quad + p^\mu \cdot (2x^2m - 4mx).
\end{aligned}
$$

Thus, we already found the coefficient A; it's the large bracket which is multiplied by γ^μ. To find B and C as well, we need to calculate them from $\tilde{B}$ and $\tilde{C}$ using the equation

$$
\tilde{B}p'^\mu + \tilde{C}p^\mu \stackrel{!}{=} B(p'^\mu + p^\mu) + C(p'^\mu - p^\mu) = (B+C)p'^\mu + (B-C)p^\mu.
$$

In our case, since $q^\mu = p'^\mu - p^\mu$,

$$
\tilde{B} = 2xm - 2xym - 2m(1-2y), \qquad \tilde{C} = 2x^2m - 4mx + 2xym + 2m(1-2y)
$$

and the equation relating B amd C with $\tilde{B}$ and $\tilde{C}$ is easily solved: From $\tilde{B} = B + C$ and $\tilde{C} = B - C$ we immediately find

$$
\begin{aligned}
B &= \frac{\tilde{B} + \tilde{C}}{2} = mx(x-1), \\
C &= \frac{\tilde{B} - \tilde{C}}{2} = m(-x^2 + 3x - 2xy + 4y - 2) = m(x-2)(1-x-2y) = m(x-2)(z-y).
\end{aligned}
$$

Hence, our result is

$$
\begin{aligned}
\widetilde{N} = \Bigg(&\left(-\frac{1}{2}l^2 + (1-y)(1-z)q^2 + (1-2x-x^2)m^2\right) \cdot \gamma^\mu + mx(x-1) \cdot (p'^\mu + p^\mu) \\
&+ m(x-2)(z-y) \cdot q^\mu\Bigg).
\end{aligned}
$$

In (>13.2.2) we concluded by the Ward identity that C, i. e. the coefficient of q^μ, must always vanish. And in fact, it does: The denominator D (see at the end of (>13.2.4)) as well as the δ-function is symmetric under exchanging the integration variables $y \leftrightarrow z$, whereas our C here is antisymmetric. Therefore, we can forget the Cq^μ-term of the numerator.

Let's now, again, switch from A and B to $\tilde{F}_1$ and $\tilde{F}_2$.[1] From the formulas in (>13.2.2) we have

$$\begin{aligned}\tilde{F}_1(q^2,l^2) &= A + 2mB = -\frac{1}{2}l^2 + (1-y)(1-z)q^2 + (1-2x-x^2)m^2 + 2m^2x(x-1)\\ &= -\frac{1}{2}l^2 + (1-y)(1-z)q^2 + (1-4x+x^2)m^2\\ \tilde{F}_2 &= -2m^2x(x-1) = 2m^2x(1-x).\end{aligned}$$

The numerator can now be given in terms of those $\tilde{F}_i$:

$$N = \bar{u}_{p'}\left(\gamma^\mu \tilde{F}_1(q^2,l^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\tilde{F}_2\right)u_p.$$

Let us summarizes our result so far (we plug in the denominator D at this point):

$$\bar{u}_{p'}\delta\Gamma^\mu u_p = 2ig^2\int d^4\bar{l}\int_0^1 dx\,dy\,dz\frac{2\delta(1-x-y-z)}{(l^2-\Delta_\Lambda+i\epsilon)^3}\bar{u}_{p'}\left(\gamma^\mu\tilde{F}_1(q^2,l^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\tilde{F}_2\right)u_p,$$

$$\begin{aligned}\text{where}\qquad \Delta_\Lambda &:= -q^2yz + m^2(1-x)^2 + x\nu^2 + x\Lambda^2,\\ \tilde{F}_1(q^2,l^2) &= -\frac{1}{2}l^2 + (1-y)(1-z)q^2 + (1-4x+x^2)m^2,\\ \tilde{F}_2 &= -2m^2x(x-1) = 2m^2x(1-x).\end{aligned}$$

By their definition, relationship between $\tilde{F}_i$ and δF_i is

$$\delta F_i = 2ig^2\int d^4\bar{l}\int_0^1 dx\,dy\,dz\frac{2\delta(1-x-y-z)}{(l^2-\Delta_\Lambda+i\epsilon)^3}\tilde{F}_i(q^2,l^2).$$

13.2.6 Wick Rotation

Our momentum integral is exactly of the form that we preannounced in section 12.4, when we introduced the concept of Wick rotation. Thus, let us perform the Wick rotation by substituting

$$l^0 = il_E^0,\qquad \vec{l} = \vec{l}_E\qquad \Longrightarrow\qquad l^2 = -l_E^2,\qquad d^4l = i\,d^4l_E,$$

where l_E is an Euclidian four-vector.

$$\bar{u}_{p'}\delta\Gamma^\mu u_p = 4g^2\int d^4\bar{l}_E\int_0^1 dx\,dy\,dz\frac{\delta(1-x-y-z)}{(l_E^2+\Delta_\Lambda)^3}\bar{u}_{p'}\left(\gamma^\mu\tilde{F}_1(q^2,-l_E^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\tilde{F}_2\right)u_p,$$

Note, that the i from the substitution of the measure combines with the i that was already present to a factor -1, which cancels the factor -1 from the denominator $(l^2-\Delta_\Lambda)^3 \to -(l_E^2+\Delta_\Lambda)^3$.

The formulas for Δ, $\tilde{F}_1$ and $\tilde{F}_2$ are given in the end of (>13.2.5).

13.2.7 Regularization with Pauli-Villars

Consider the result of (>13.2.6) assuming that we had not introduced a Pauli-Villars regulator Λ. That is, use $\Delta := \Delta_{\Lambda=0}$ instead of Δ_Λ. Focus on the momentum integration variable l_E. It appears in the denominator $(l_E^2+\Delta)^3$ and in the numerator term $\tilde{F}_1(q^2,-l_E^2) = l_E^2/2 + (\text{terms independent of } l)$. Thus, we have two types of momentum integrals:

$$\int d^4\bar{l}_E\frac{1}{(l_E^2+\Delta)^3}\qquad\text{and}\qquad\int d^4\bar{l}_E\frac{l_E^2}{(l_E^2+\Delta)^3},$$

[1] We use $\tilde{F}_i$ here instead of F_i, since we introduced F_i as the formfactors when expanding Γ^μ in γ^μ and $\sigma^{\mu\nu}q_\nu$. However, here we only expand the numerator N. In contrast to F_i, the formfactors $\tilde{F}_i$ can also depend on l^2.

where $\Delta = -q^2yz + m^2(1-x)^2 + xv^2$. Using the formulas of Pauli-Villars regularization in section 12.7, we see that the first of these two integrals falls into the category $n > 2$ of the first integral given in section 12.7. There, we stated that renormalization is not necessary for this term, since it is convergent. However, the second of the integrals above falls into the category $n = 3$ of the second integral given in section 12.7. That is, it is divergent, unless we introduce a Pauli-Villars regulator. Hence, let's do just that. Now would be the point to go through the whole calculation of the previous section again, using a photon propagator with a Pauli-Villars regulator $\Lambda \neq 0$. Fortunately, we have already included it in the beginning.

Well, the Pauli-Villars regularization does not simply tell us to introduce a photon mass Λ into the propagator – it tells us, that we need to *subtract* a massive photon propagator from the usual one. That is, in Pauli-Villars regularization, $\delta\Gamma^\mu$ gets two contributions, one from a massless and one from a massive photon. Correspondingly, the momenta integrals above will turn into

$$\int d^4\bar{l}_E \frac{1}{(l_E^2+\Delta)^3} \to \int d^4\bar{l}_E \left(\frac{1}{(l_E^2+\Delta)^3} - \frac{1}{(l_E^2+\Delta_\Lambda)^3}\right) = \frac{\Omega_4 b_3}{(2\pi)^4}\left(\frac{1}{\Delta} - \frac{1}{\Delta_\Lambda}\right) = \frac{1}{2(4\pi)^2}\frac{1}{\Delta}$$

and

$$\int d^4\bar{l}_E \frac{l_E^2}{(l_E^2+\Delta)^3} \to \int d^4\bar{l}_E \left(\frac{l_E^2}{(l_E^2+\Delta)^3} - \frac{l_E^2}{(l_E^2+\Delta_\Lambda)^3}\right) = \frac{\Omega_4}{(2\pi)^4}\cdot\frac{1}{2}\ln\Delta_\Lambda/\Delta = \frac{1}{(4\pi)^2}\ln\Delta_\Lambda/\Delta.$$

where we plugged in $\Omega_4 = 2\pi^2$ and $b_3 = 1/4$. In the first integral, we could apply the limit $\Lambda \to \infty$.

Let's apply these discoveries to our loop integral from (>13.2.6). Let's start with the $\tilde{F}_1$-term and let

$$\tilde{\mathcal{F}} \coloneqq (1-y)(1-z)q^2 + (1-4x+x^2)m^2 \qquad \Longrightarrow \qquad \tilde{F}_1(q^2,-l_E^2) = l_E^2/2 + \tilde{\mathcal{F}}$$

be an abbreviation for the l_E-independent terms in $\tilde{F}_1$ from the very end of (>13.2.5). Then, introducing in the "$\to$"-step the second massive photon propagator, we find

$$\begin{aligned}
\bar{u}_{p'}\delta\Gamma^\mu u_p &= 4g^2\int d^4\bar{l}_E\int_0^1 dx\,dy\,dz\frac{\delta(1-x-y-z)}{(l_E^2+\Delta)^3}\bar{u}_{p'}\left(\gamma^\mu\tilde{F}_1(q^2,-l_E^2)\right)u_p + \tilde{F}_2\text{-term}\\
&\to 4g^2\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z)\\
&\qquad\qquad \bar{u}_{p'}\gamma^\mu\int d^4\bar{l}_E\left(\frac{1}{(l_E^2+\Delta)^3} - \frac{1}{(l_E^2+\Delta_\Lambda)^3}\right)\left(l_E^2/2+\tilde{\mathcal{F}}\right)\,u_p + \tilde{F}_2\text{-term}\\
&= 4g^2\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z)\;\bar{u}_{p'}\gamma^\mu\left(\frac{1}{2}\frac{1}{(4\pi)^2}\ln\Delta_\Lambda/\Delta + \tilde{\mathcal{F}}\frac{1}{2(4\pi)^2}\frac{1}{\Delta}\right)u_p + \tilde{F}_2\text{-term}\\
&= \frac{2g^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z)\;\bar{u}_{p'}\gamma^\mu\left(\ln\Delta_\Lambda/\Delta + \frac{\tilde{\mathcal{F}}}{\Delta}\right)u_p + \tilde{F}_2\text{-term}\\
&= \bar{u}_{p'}\gamma^\mu\,\delta F_1(q^2)\,u_p + \tilde{F}_2\text{-term}.
\end{aligned}$$

$\tilde{F}_2$ does not depend on l_E^2 at all (see (>13.2.5)), hence its term becomes

$$
\begin{aligned}
\bar{u}_{p'}\delta\Gamma^\mu u_p &= \tilde{F}_1\text{-term} + 4g^2 \int d^4\bar{l}_E \int_0^1 dx\, dy\, dz \frac{\delta(1-x-y-z)}{(l_E^2+\Delta)^3} \bar{u}_{p'} \left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\tilde{F}_2\right) u_p \\
&\to \tilde{F}_1\text{-term} + 4g^2 \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z) \\
&\qquad\qquad \bar{u}_{p'} \frac{i\sigma^{\mu\nu}q_\nu}{2m} \int d^4\bar{l}_E \left(\frac{1}{(l_E^2+\Delta)^3} - \frac{1}{(l_E^2+\Delta_\Lambda)^3}\right) \tilde{F}_2 u_p \\
&= \tilde{F}_1\text{-term} + 4g^2 \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z)\ \bar{u}_{p'} \frac{i\sigma^{\mu\nu}q_\nu}{2m} \left(\frac{1}{2(4\pi)^2}\frac{1}{\Delta}\tilde{F}_2\right) u_p \\
&= \tilde{F}_1\text{-term} + \frac{2g^2}{(4\pi)^2} \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z)\ \bar{u}_{p'} \frac{i\sigma^{\mu\nu}q_\nu}{2m} \left(\frac{1}{\Delta}\tilde{F}_2\right) u_p \\
&= \tilde{F}_1\text{-term} + \bar{u}_{p'} \frac{i\sigma^{\mu\nu}q_\nu}{2m} \delta F_2(q^2)\, u_p.
\end{aligned}
$$

Obviously, since the $\tilde{F}_2$-term is not divergent at all, we could have skipped the step "$\to$" where we introduced the Pauli-Villars regulator, and we would have ended up with the same result.

We can now extract the expressions for δF_1 and δF_2 from the two computations above:

$$
\delta F_1 = \frac{2g^2}{(4\pi)^2} \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z) \left(\ln\frac{x\Lambda^2}{\Delta} + \frac{(1-y)(1-z)q^2 + (1-4x+x^2)m^2}{\Delta}\right),
$$

$$
\delta F_2 = \frac{2g^2}{(4\pi)^2} \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z)\ \frac{2m^2x(1-x)}{\Delta},
$$

where we plugged in $\tilde{\mathcal{F}}$ from above and $\tilde{F}_2$ from (>13.2.5). Also, since we are going to take the limit $\Lambda \to \infty$ in the end, we could also replace $\Delta_\Lambda \to x\Lambda^2$. Also, note for the prefactor that

$$
\frac{2g^2}{(4\pi)^2} = \frac{2\cdot 4\pi\alpha}{(4\pi)^2} = \frac{\alpha}{2\pi}.
$$

Δ is the one without Pauli-Villars regulator, that is $\Delta = -q^2yz + (1-x)^2m^2 + x\nu^2$.

13.2.8 Infrared Divergence of the first Form Factor

The first form factor contains a logarithm term and one term $\sim 1/\Delta$. To make the infrared divergence apparent, consider the special case $q^2 = 0$ for the latter term only in the limit $\nu = 0$:

$$
\begin{aligned}
\frac{\alpha}{2\pi}\int_0^1 dx\, dy\, dz\, \delta(1-x-y-z)\frac{(1-4x+x^2)m^2}{\Delta(q^2=0)} &= \int_0^1 dx\,(1-x)\frac{(1-4x+x^2)m^2}{(1-x)^2m^2} \\
= \int_0^1 dx\ \frac{(1-x)^2-2x}{1-x} = \int_0^1 dx\ \left(1-x-\frac{2x}{1-x}\right) &= \frac{1}{2} - \int_0^1 dx\ \frac{2x}{1-x}.
\end{aligned}
$$

This integral is clearly divergent at the upper boundary.

Note, that the integration over dy, dz and the δ-function lead to a factor $1-x$. Why is this the case? To understand this, let's define a kind of a Heaviside function,

$$
\theta(z,a,b) := \begin{cases} 1, & a \le z \le b \\ 0, & \text{else} \end{cases}.
$$

Then, consider

$$
\begin{aligned}
&\int_0^1 dx \int_0^1 dy \int_0^1 dz\ f(x,y,z)\, \delta(1-x-y-z) \\
&\qquad = \int_0^1 dx \int_0^1 dy \int_{-\infty}^{\infty} dz\ f(x,y,z)\, \theta(z,0,1)\, \delta(1-x-y-z) \\
&\qquad = \int_0^1 dx \int_0^1 dy\, f(x,y,1-x-y)\, \theta(1-x-y,0,1) \\
&\qquad = \int_0^1 dx \int_0^1 dy\, f(x,y,1-x-y)\, \theta(y,0,1-x) = \int_0^1 dx \int_0^{1-x} dy\, f(x,y,1-x-y).
\end{aligned}
$$

In one step in the middle, we used $\theta(1-x-y,0,1) = \theta(y,0,1-x)$. That is okay, if $0 < x < 1$, because then $0 \le 1-x-y \le 1$ is equivalent to $0 \le y \le 1-x$.

From this general case, we easily can find the special cases

$$
\int_0^1 dx \int_0^1 dy \int_0^1 dz\ f(x,y)\, \delta(1-x-y-z) = \int_0^1 dx \int_0^{1-x} dy\, f(x,y),
$$

$$
\int_0^1 dx \int_0^1 dy \int_0^1 dz\ f(x)\, \delta(1-x-y-z) = \int_0^1 dx \int_0^{1-x} dy\, f(x) = \int_0^1 dx\, (1-x)\, f(x).
$$

13.2.9 Explicit Result for the second Form Factor at Zero Momentum

Recall that $F_2(q^2) = 0 + \delta F_2(q^2) + \cdots$. Using the formula for $\delta F_2(q^2)$ derived in 13.2.7, we find in the case $q^2 = 0$ (since δF_2 is not infrared divergent, we can immediately set $\nu = 0$)

$$
\begin{aligned}
F_2(q^2) = \delta F_2(0) &= \frac{\alpha}{2\pi} \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z)\, \frac{2m^2 x(1-x)}{m^2(1-x)^2} \\
&= \frac{\alpha}{2\pi} \int_0^1 dx\, (1-x)\, \frac{2m^2 x(1-x)}{m^2(1-x)^2} = \frac{\alpha}{2\pi} \int_0^1 dx\, 2x = \frac{\alpha}{2\pi},
\end{aligned}
$$

where we used $\int_0^1 dx\, dy\, dz\, \delta(1-x-y-z) f(x) = \int_0^1 dx\, (1-x) f(x)$ from (>13.2.8).

13.2.10 Vertex Renormalization Factor – Relation to the Form Factor

Note that the prefactor of F_2 contains a q_ν, such that

$$
\Gamma^\mu(0) = \gamma^\mu F_1(0)
$$

and hence,

$$
\gamma^\mu = Z_1\, \Gamma^\mu(0) = (1+\delta_1)\big(\gamma^\mu + \gamma^\mu \delta F_1(0) + \mathcal{O}(\alpha^2)\big) = \gamma^\mu \left(1 + \delta F_1(0) + \delta_1^{(2)} + \mathcal{O}(\alpha^2)\right)
$$

$$
\Leftrightarrow \qquad \delta_1^{(2)} = -\delta F_1(0).
$$

13.2.11 Vertex Renormalization Factor – Explicit Computation

Since setting $q^2 = 0$ inside $\delta F_1(q^2)$ also makes the integrand independent of the Feynman parameters y, z, we can simplify the integral for $q^2 = 0$ as follows:

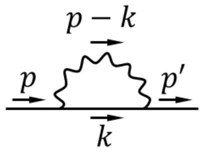
$$
\begin{aligned}
\delta_1^{(2)} &= -\delta F_1(0) \\
&= -\frac{\alpha}{2\pi} \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z) \left(\ln \frac{x\Lambda^2}{(1-x)^2 m^2 + x\nu^2} + \frac{(1-4x+x^2)m^2}{(1-x)^2 m^2 + x\nu^2} \right) \\
&= -\frac{\alpha}{2\pi} \int_0^1 dx\, (1-x) \left(\ln \frac{x\Lambda^2}{(1-x)^2 m^2 + x\nu^2} + \frac{(1-4x+x^2)m^2}{(1-x)^2 m^2 + x\nu^2} \right).
\end{aligned}
$$

Note that we used $\int_0^1 dx\, dy\, dz\, f(x)\, \delta(1-x-y-z) = \int_0^1 dx\, (1-x) f(x)$ from (>13.2.8).

13.3 The Electron Self-Energy

13.3.1 1PI to Leading Order

To order α, $-i\Sigma$ is the *amputated* (that is, without external propagators/spinors) diagram



Let us call this diagram alone $-i\Sigma^{(2)}$. Luckily, the computation of this diagram is a lot easier than the vertex correction.

FEYNMAN RULES (from section 8.2):

From Feynman rules, we find (let us again provisionally put in a Pauli-Villars regulator Λ and a small photon mass ν)

$$\begin{aligned}-i\Sigma^{(2)}(p) &= \int d^4\bar{k}\;\; ig\gamma^\mu\; \frac{i}{\not{k}-m_0+i\epsilon}\;\; ig\gamma^\nu\; \frac{-i\eta_{\mu\nu}}{(p-k)^2-\nu^2-\Lambda^2+i\epsilon}\\ &= -g^2\int d^4\bar{k}\;\;\gamma^\mu \frac{\not{k}+m}{k^2-m_0^2+i\epsilon}\gamma_\mu \frac{1}{(p-k)^2-\nu^2-\Lambda^2+i\epsilon}.\end{aligned}$$

FEYNMAN PARAMETERS (from section 12.2):

First, we introduce Feynman parameters to combine the denominators according to section 12.2:

$$\begin{aligned}\frac{1}{k^2-m_0^2+i\epsilon}\frac{1}{(p-k)^2+i\epsilon} &= \int_0^1 dx\,\frac{1}{\left((k^2-m_0^2+i\epsilon)(1-x)+((p-k)^2-\nu^2-\Lambda^2+i\epsilon)x\right)^2}\\ &= \int_0^1 dx\,\frac{1}{(k^2-m_0^2(1-x)+(p^2-2p\cdot k)x-x\nu^2-x\Lambda^2+i\epsilon)^2}\\ &= \int_0^1 dx\,\frac{1}{(l^2+2x\,p\cdot l+x^2p^2-m_0^2(1-x)+(p^2-2p\cdot l-2x\,p^2)x-x\nu^2-x\Lambda^2+i\epsilon)^2}\\ &= \int_0^1 dx\,\frac{1}{(l^2+x(1-x)p^2-m_0^2(1-x)-x\nu^2-x\Lambda^2+i\epsilon)^2} = \int_0^1 dx\,\frac{1}{(l^2-\Delta_\Lambda+i\epsilon)^2},\end{aligned}$$

where we introduced $l := k - xp$, plugged in $k = l + xp$ and used

$$\Delta_\Lambda := -x(1-x)p^2+(1-x)m_0^2+x\nu^2+x\Lambda^2.$$

DIRAC ALGEBRA (from section 12.3):

In the numerator we use $\gamma^\mu\not{k}\gamma_\mu = -2\not{k} = -2\not{l}-2x\not{p}$ and $\gamma^\mu\gamma_\mu = 4$. By symmetry, the numerator-term proportional to $\not{l}$ vanishes after integration and we are left with

$$-i\Sigma^{(2)}(p) = -g^2\int d^4\bar{l}\int_0^1 dx\,\frac{-2x\not{p}+4m_0}{(l^2-\Delta+i\epsilon)^2}.$$

WICK ROTATION (from section 12.4):

Performing a Wick rotation $l^0 = il_E^0, \vec{l} = \vec{l}_E$, such that $d^4l = i\,d^4l_E$ and $l^2 = -l_E^2$, we find

$$-i\Sigma^{(2)}(p) = -ig^2\int d^4\bar{l}_E\int_0^1 dx\,\frac{-2x\not{p}+4m_0}{(l_E^2+\Delta)^2}.$$

PAULI-VILLARS REGULARIZATION (from section 12.7):

When we go to spherical coordinates, it is obvious that the momentum integral is divergent. We regularize with Pauli-Villars by subtracting a heavy mass photon propagator:

$$\frac{1}{(p-k)^2+i\epsilon} \quad \longrightarrow \quad \frac{1}{(p-k)^2+i\epsilon} - \frac{1}{(p-k)^2-\Lambda^2+i\epsilon}.$$

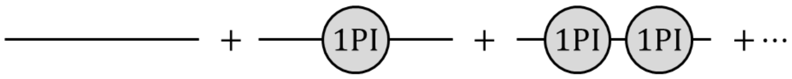
Thus, using $\Delta \coloneqq \Delta_{\Lambda=0}$ as well as the integration formulas from section 12.7

$$\begin{aligned}
-i\Sigma^{(2)}(p) &= -ig^2 \int_0^1 dx\, (-2x\not{p} + 4m_0) \int d^4\bar{l}_E \left(\frac{1}{(l_E^2+\Delta)^2} - \frac{1}{(l_E^2+\Delta_\Lambda)^2} \right) \\
&= -ig^2 \int_0^1 dx\, (-2x\not{p}+4m_0) \frac{\Omega_4}{(2\pi)^4} \cdot \frac{1}{2} \ln \Delta_\Lambda/\Delta = -i\frac{2g^2}{(4\pi)^2} \int_0^1 dx\, (-x\not{p}+2m_0) \ln \Delta_\Lambda/\Delta \\
&= -i\frac{\alpha}{2\pi} \int_0^1 dx\, (-x\not{p}+2m_0) \ln \frac{x\Lambda^2}{\Delta},
\end{aligned}$$

where we used $\Omega_4 = 2\pi^2, g = \sqrt{4\pi\alpha}$ as well as the fact that we are interested into the limit $\Lambda \to \infty$, so that we can replace $\Delta_\Lambda \to x\Lambda^2$. Since $p^2 = \not{p}^2$, we will from now on give Σ_p as a function of $\not{p}$ rather than p; that is, $\Sigma^{(2)}(p) \to \Sigma^{(2)}(\not{p})$.

13.3.2 Adding up 1PIs

We now want to examine what we can draw graphically as

$\mathrm{FT}\langle\Omega|\mathcal{T}\, \psi(x)\bar{\psi}(y)|\Omega\rangle =$ 

We already talked about the fact that the mass in the Lagrangian m_0 differs from the actual particle mass m. We will now impose that particle masses m are poles of the propagators. Since the propagator on the left-hand side accounts for the whole physical particle, including self-interactions, we take m as its pole: $i/(\not{p}-m)$. The expression on the right-hand side is what we get from Feynman rules, thus we use the mass m_0 here. We can turn it into a geometric series and find

$$\begin{aligned}
&\mathrm{FT}\,\langle\Omega|\mathcal{T}\,\psi(x)\bar{\psi}(y)|\Omega\rangle \\
&\quad = \frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}(-i\Sigma)\frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}(-i\Sigma)\frac{i}{\not{p}-m_0}(-i\Sigma)\frac{i}{\not{p}-m_0} + \cdots \\
&\quad = \frac{i}{\not{p}-m_0} \sum_{n=0}^{\infty} \left((-i\Sigma)\frac{i}{\not{p}-m_0} \right)^n = \frac{i}{\not{p}-m_0} \frac{1}{1-(-i\Sigma)\frac{i}{\not{p}-m_0}} = \frac{i}{\not{p}-m_0-\Sigma}.
\end{aligned}$$

13.3.3 Structure of the Interacting Propagator

In section 7.3, we derived the Källén-Lehman spectral representation in the form

$$\langle\Omega|\mathcal{T}\,\psi(x)\bar{\psi}(y)|\Omega\rangle = Z_2\widetilde{D}_F(x-y,m) + \int_{(2m)^2}^{\infty} dM^2\,\rho(M^2)\,\widetilde{D}_F(x-y,M),$$

where we have already turned the equation into its equivalent form for the application to fermion fields. $\widetilde{D}_F(x-y,m)$ is the Feynman propagator with mass m (in the fermion case, it depends on m rather than m^2 only). Its Fourier transform is

$$\mathrm{FT}\,\widetilde{D}_F(x-y,m) = \frac{i}{\not{p}-m}.$$

The terms in the integral are obviously finite at $p = m$. Thus, we find

$$\mathrm{FT}\,\langle\Omega|\mathcal{T}\,\psi(x)\bar{\psi}(y)|\Omega\rangle = \frac{Z_2 i}{\not{p}-m} + \text{terms finite at } \not{p} = m.$$

Since we just computed

$$\text{FT}\,\langle\Omega|\mathcal{T}\,\psi(x)\bar{\psi}(y)|\Omega\rangle = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})},$$

we can equate our results *in the limit* $\not{p} \to m$ as follows:

$$\frac{i}{\not{p} - m_0 - \Sigma(\not{p})} = \frac{Z_2 i}{\not{p} - m}.$$

13.3.4 Corrections to the Mass and the Field-Strength Renormalization

To leading order perturbation theory, the full propagator should give us back the non-interacting propagator:

$$\frac{iZ_2}{\not{p} - m} = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})} = \frac{i}{\not{p} - m_0} + \mathcal{O}(\alpha).$$

Thus, up to first order, we naturally can take $Z_2 = 1$ and $m = m_0$ or $\Sigma(\not{p}) = 0$. But beyond the first order, the $\Sigma(\not{p}) \neq 0$ will shift the pole of the propagator away from m_0. Thus, already to next leading order, we need to correct also our mass $m = m_0 + \delta m$.

So m should be defined as the pole of the full propagator: In analogy to $1/(\not{p} - m)$ having a pole $\not{p} = m$ or $\not{p} - m|_{\not{p}=m} = 0$, we say that $1/\big(\not{p} - m_0 - \Sigma(p)\big)$ has a pole at the m which fulfills the equation

$$\not{p} - m_0 - \Sigma(\not{p})|_{\not{p}=m} = 0 \qquad \Leftrightarrow \qquad m = m_0 + \Sigma(\not{p} = m).$$

If we do a Taylor expansion of the denominator close to the pole $\not{p} = m$, we find[1]

$$\begin{aligned} \not{p} - m_0 - \Sigma(m) &\approx \underbrace{\big(m - m_0 - \Sigma(m)\big)}_{=0} + \frac{\partial}{\partial \not{p}}\big(\not{p} - m_0 - \Sigma(\not{p})\big)\Big|_{\not{p}=m}(\not{p} - m) \\ &= \left(1 - \frac{\partial \Sigma(\not{p})}{\partial \not{p}}\Big|_{\not{p}=m}\right)(\not{p} - m). \end{aligned}$$

If we plug this in into the equation above, we find

$$\frac{iZ_2}{\not{p} - m} = \frac{i}{(\not{p} - m)\left(1 - \frac{\partial \Sigma(\not{p})}{\partial \not{p}}\Big|_{\not{p}=m}\right)} \qquad \Rightarrow \qquad Z_2 = \left(1 - \frac{\partial \Sigma(\not{p})}{\partial \not{p}}\Big|_{\not{p}=m}\right)^{-1}.$$

13.3.5 Correction to the Mass

Using $m = m_0 + \Delta m^{(2)} + \mathcal{O}(\alpha^2)$ and the order-α result of $\Sigma_{\not{p}}$ (we can replace $m_0 = m + \mathcal{O}(\alpha)$) we find (recall that $p^2 = \not{p}^2$)

$$\begin{aligned} \Delta m^{(2)} = \Sigma^{(2)}(\not{p} = m) &= \frac{\alpha}{2\pi}\int_0^1 dx\,(-xm + 2m)\ln\frac{x\Lambda^2}{(1-x)m^2 - x(1-x)m^2 + x\nu^2} \\ &= \frac{\alpha}{2\pi}\int_0^1 dx\,(2 - x)m\ln\frac{x\Lambda^2}{(1-x)^2 m^2 + x\nu^2}. \end{aligned}$$

13.3.6 Correction to the Electron Field Strength Renormalization

Using $Z = 1 + \delta_2^{(2)} + \mathcal{O}(\alpha^2)$ and the order-α result of $\Sigma_{\not{p}}$, we find

$$Z_2 = \left(1 - \frac{\partial \Sigma(\not{p})}{\partial \not{p}}\Big|_{\not{p}=m}\right)^{-1} = 1 + \frac{\partial \Sigma(\not{p})}{\partial \not{p}}\Big|_{\not{p}=m} + \mathcal{O}(\alpha) = 1 + \frac{\partial \Sigma^{(2)}(\not{p})}{\partial \not{p}}\Big|_{\not{p}=m} + \mathcal{O}(\alpha).$$

[1] Being physicists and not mathematicians, let's avoid to be disturbed too much about what it means to take a derivative with respect to a matrix $\not{p}$.

Thus,

$$
\begin{aligned}
\delta_2^{(2)} = \frac{\partial \Sigma^{(2)}(\not{p})}{\partial \not{p}}\bigg|_{\not{p}=m} &= \frac{\alpha}{2\pi}\frac{\partial}{\partial \not{p}}\int_0^1 dx\,(-x\not{p}+2m)\ln\frac{x\Lambda^2}{-x(1-x)\not{p}^2+(1-x)m^2+x\nu^2}\bigg|_{\not{p}=m} \\
&= \frac{\alpha}{2\pi}\int_0^1 dx\bigg(-x\ln\frac{x\Lambda^2}{-x(1-x)\not{p}^2+(1-x)m^2+x\nu^2} \\
&\qquad - \frac{(-x\not{p}+2m)\big(-2\not{p}x(1-x)\big)}{-x(1-x)\not{p}^2+(1-x)m^2+x\nu^2}\bigg)\bigg|_{\not{p}=m} \\
&= \frac{\alpha}{2\pi}\int_0^1 dx\bigg(-x\ln\frac{x\Lambda^2}{(1-x)^2m^2+x\nu^2}+2(2-x)\frac{x(1-x)m^2}{(1-x)^2m^2+x\nu^2}\bigg),
\end{aligned}
$$

where we could set $m_0 = m + \mathcal{O}(\alpha)$ again.

13.4 The Photon Self-Energy (Vacuum Polarization)

13.4.1 General Form of the Photon 1PI

The only tensors that can appear in $\Pi^{\mu\nu}(q)$ are $\eta^{\mu\nu}$ and $q^\mu q^\nu$ (tensors with γ-matrices, like $\gamma^\mu\gamma^\nu$, cannot appear, since then $\Pi^{\mu\nu}$ would be a matrix itself, which it isn't). Using the Ward identity from section 11.5, we know that $q_\mu \Pi^{\mu\nu}(q) = 0$. Writing $\Pi^{\mu\nu}(q)$ in full generality, we can deduce

$$
\Pi^{\mu\nu}(q) = A\eta^{\mu\nu} + Bq^\mu q^\nu \quad \Longrightarrow \quad q_\mu \Pi^{\mu\nu}(q) = Aq^\nu + Bq^2q^\nu \overset{!}{=} 0 \quad \Longleftrightarrow \quad B = -A/q^2,
$$

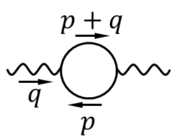
where A, B are in general functions of q^2. Thus, we will write

$$
\Longrightarrow \qquad \Pi^{\mu\nu}(q) = (\eta^{\mu\nu} - q^\mu q^\nu/q^2)A = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2).
$$

Note that lowest order diagram within the 1PI is the one electron loop diagram. Thus, in any diagram that contribute to $\Pi^{\mu\nu}(q)$ massive particles are involved. Therefore, we expect that $\Pi^{\mu\nu}(q)$ or $\Pi(q^2)$ respectively does not have a pole at $q^2 = 0$. That is, $\Pi(q^2)$ is *regular* at $q^2 = 0$.

13.4.2 Computation of the Electron Loop – Feynman Rules

Using Feynman rules to translate the diagram



into mathematics, we find

$$
\begin{aligned}
i\Pi^{\mu\nu}_{(2)}(q) &= (-1)\int d^4\bar{p}\ \mathrm{Tr}\left(ig\gamma^\mu\frac{i}{\not{p}-m+i\epsilon}ig\gamma^\nu\frac{i}{\not{p}+\not{q}-m+i\epsilon}\right) \\
&= -g^2\mu^\epsilon\int d^d\bar{p}\ \mathrm{Tr}\left(\gamma^\mu\frac{\not{p}+m}{p^2-m^2+i\epsilon}\gamma^\nu\frac{\not{p}+\not{q}+m}{(p+q)^2-m^2+i\epsilon}\right).
\end{aligned}
$$

This time, we want to use dimensional regularization instead of Pauli-Villars regularization. Thus, we wrote down $d^d\bar{p}$ and $g \to g\mu^{\epsilon/2}$ (see section 12.6).

13.4.3 Computation of the Electron Loop – Feynman Parameter

Using a Feynman parameter, we can write

$$
\begin{aligned}
&\frac{1}{(p^2-m^2+i\epsilon)((p+q)^2-m^2+i\epsilon)} \\
&\qquad = \int_0^1 dy \frac{1}{\big(y(p^2-m^2+i\epsilon)+(1-y)((p+q)^2-m^2+i\epsilon)\big)^2} \\
&\qquad = \int_0^1 dy \frac{1}{((p+q)^2-m^2-y(2p\cdot q+q^2)+i\epsilon)^2} \\
&\qquad = \int_0^1 dy \frac{1}{(p^2+(1-y)(2p\cdot q+q^2)-m^2+i\epsilon)^2} \\
&\qquad = \int_0^1 dx \frac{1}{(p^2+x(2p\cdot q+q^2)-m^2+i\epsilon)^2} \\
&\qquad = \int_0^1 dx \frac{1}{(p^2+2xp\cdot q+x^2q^2+xq^2-x^2q^2-m^2+i\epsilon)^2} \\
&\qquad = \int_0^1 dx \frac{1}{(l^2+x(1-x)q^2-m^2+i\epsilon)^2} = \int_0^1 dx \frac{1}{(l^2-\Delta+i\epsilon)^2}
\end{aligned}
$$

Here, we substituted $y = 1-x \Longrightarrow \int_0^1 dy = -\int_1^0 dx = \int_0^1 dx$ and introduced $l \coloneqq p + xq$. We have

$$\Delta = -x(1-x)q^2 + m^2.$$

13.4.4 Computation of the Electron Loop – Dirac Algebra

Let us now tackle the numerator of $i\Pi^{\mu\nu}$,

$$N \coloneqq \mathrm{Tr}\big(\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}+\not{q}+m)\big)$$

We now want to regularize the momentum integral by dimensional regularization, so we need the general formulas for the Dirac algebra from 12.3. Recall, that the trace over an odd number of γ-matrices vanishes and that

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \tilde{4}\eta^{\mu\nu}, \qquad \mathrm{Tr}\,\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma = \tilde{4}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}).$$

By $\tilde{4}$, we mean $\tilde{4} \coloneqq \mathrm{Tr}\,\mathbb{I}_d$ (see section 12.3). In the end, we can simply turn $\tilde{4}$ into $\mathrm{Tr}\,\mathbb{I}_4 = 4$. Thus, the numerator takes on the form

$$
\begin{aligned}
N &= \mathrm{Tr}\,\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}+\not{q}+m) \\
&= \mathrm{Tr}\big(\gamma^\mu\not{p}\gamma^\nu(\not{p}+\not{q}+m) + m\gamma^\mu\gamma^\nu(\not{p}+\not{q}+m)\big) = \mathrm{Tr}\big(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma p_\rho(p+q)_\sigma + m^2\gamma^\mu\gamma^\nu\big) \\
&= \tilde{4}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu})p_\rho(p+q)_\sigma + \tilde{4}m^2\eta^{\mu\nu} \\
&= \tilde{4}(p^\mu(p+q)^\nu - \eta^{\mu\nu}p\cdot(p+q) + p^\nu(p+q)^\mu + m^2\eta^{\mu\nu}) \\
&= \tilde{4}\big(p^\mu(p+q)^\nu + p^\nu(p+q)^\mu - \eta^{\mu\nu}(p\cdot(p+q) - m^2)\big).
\end{aligned}
$$

Having substituted $l = p + xq$ in the denominator, let's see how the numerator is changed by this substitution. Of course, the integration will then be over $d^d\bar{l}$ instead of $d^d\bar{p}$. Note, that we immediately drop terms linear in l^μ, as they are going to vanish after integration by symmetry (see section 12.3). There are three terms in the numerator, the first to of which are equivalent under $\mu \leftrightarrow \nu$. Thus, we only need to consider two substantially different terms (we plug in $p = l - xq$):

$$p^\mu(p+q)^\nu = (l-xq)^\mu(l+(1-x)q)^\nu = l^\mu l^\nu - x(1-x)q^\mu q^\nu + (\text{terms linear in } l),$$

$$
\begin{aligned}
-\eta^{\mu\nu}(p\cdot(p+q) - m^2) &= -\eta^{\mu\nu}\big((l-xq)\cdot(l+(1-x)q) - m^2\big) \\
&= -\eta^{\mu\nu}(l^2 - x(1-x)q^2 - m^2) + (\text{terms linear in } l).
\end{aligned}
$$

Thus, the total numerator in terms of l instead of p reads

$$
\begin{aligned}
N/\tilde{4} &= 2l^\mu l^\nu - \eta^{\mu\nu}l^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(x(1-x)q^2 + m^2) + (\text{terms linear in } l) \\
&= -(1-2/d)\eta^{\mu\nu}\, l^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(x(1-x)q^2 + m^2),
\end{aligned}
$$

where we dropped the linear terms and replaced $l^\mu l^\nu \to l^2\eta^{\mu\nu}/d$, according to section 12.3.

Thus, we have arrived at

$$i\Pi^{\mu\nu}_{(2)} = -\tilde{4}g^2\mu^\epsilon \int_0^1 dx \int d^d\bar{l}\, \frac{-(1-2/d)\eta^{\mu\nu}\, l^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(x(1-x)q^2 + m^2)}{(l^2 - \Delta + i\epsilon)^2}.$$

13.4.5 Computation of the Electron Loop – Wick Rotation

After performing the Wick rotation from section 12.4, $l^0 = il_E^0, \vec{l} = \vec{l}_E, d^d\bar{l} = i\, d^d\bar{l}_E, l^2 = -l_E^2$, we find

$$i\Pi^{\mu\nu}_{(2)} = -\tilde{4}ig^2\mu^\epsilon \int_0^1 dx \int d^d\bar{l}_E\, \frac{(1-2/d)\eta^{\mu\nu}\, l_E^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(x(1-x)q^2 + m^2)}{(l_E^2 + \Delta)^2},$$

where $\quad \Delta = -x(1-x)q^2 + m^2.$

13.4.6 Computation of the Electron Loop – Momentum Integral in d Dimensions

The numerator has one term $\sim l_E^2$ and one constant term (with respect to l^μ). Using the formulas from section 12.6 in the case $n = 2$, we find (using $\Gamma(2) = 1$)

$$\int d^d\bar{l}_E\, \frac{1}{(l_E^2 + \Delta)^2} = \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{\epsilon/2}},$$

$$\int d^d\bar{l}_E\, \frac{l_E^2}{(l_E^2 + \Delta)^2} = \frac{d}{2} \frac{\Gamma(-1+\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{-1+\epsilon/2}}.$$

The term with the l_E^2-term in the numerator of (>13.4.5) will then read, using $d = 4 - \epsilon$,

$$\begin{aligned}
\int d^d\bar{l}_E\, \frac{(1-2/d)\eta^{\mu\nu}\, l_E^2}{(l_E^2 + \Delta)^2} &= \left(\left(1 - \frac{2}{d}\right)\eta^{\mu\nu}\right)\left(\frac{d}{2}\frac{\Gamma(-1+\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{-1+\epsilon/2}}\right) \\
&= \left(\frac{4-\epsilon}{2} - 1\right)\eta^{\mu\nu} \frac{\Gamma(-1+\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \frac{\Delta}{\Delta^{\epsilon/2}} \\
&= -\left(-1 + \frac{\epsilon}{2}\right)\eta^{\mu\nu} \frac{\Gamma(-1+\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \frac{\Delta}{\Delta^{\epsilon/2}} \\
&= \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{\epsilon/2}} \left(\eta^{\mu\nu}(x(1-x)q^2 - m^2)\right)
\end{aligned}$$

where in the last step we used $x\Gamma(x) = \Gamma(x+1)$ and plugged in $\Delta = -x(1-x)q^2 + m^2$ in the numerator. Putting those results together, we find

$$\begin{aligned}
i\Pi^{\mu\nu}_{(2)} &= -\tilde{4}ig^2\mu^\epsilon \int_0^1 dx \left(\frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \frac{1}{\Delta^{\epsilon/2}}\right) \\
&\qquad \left(\eta^{\mu\nu}(x(1-x)q^2 - m^2) - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(x(1-x)q^2 + m^2)\right) \\
&= -\frac{\tilde{4}ig^2}{(4\pi)^2} \int_0^1 dx \left(\frac{\Gamma(\epsilon/2)}{(4\pi)^{-\epsilon/2}} \frac{\mu^\epsilon}{\Delta^{\epsilon/2}}\right) \left(2x(1-x)\right)(q^2\eta^{\mu\nu} - q^\mu q^\nu) \\
&= (q^2\eta^{\mu\nu} - q^\mu q^\nu)\, i\Pi^{(2)}(q^2),
\end{aligned}$$

which has exactly the structure prophesied. Using the expansions $\Gamma(a\epsilon) = 1/a\epsilon - \gamma + \mathcal{O}(\epsilon)$ as well as $\Delta^{a\epsilon} = 1 + a\epsilon \ln \Delta + \mathcal{O}(\epsilon^2)$ (also for μ and 4π) from section 12.6, we find

$$\begin{aligned}
\left(\frac{\Gamma(\epsilon/2)}{(4\pi)^{-\epsilon/2}} \frac{\mu^\epsilon}{\Delta^{\epsilon/2}}\right) &= \Gamma(\epsilon/2)\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon/2} = \left(\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right)\left(1 + \frac{\epsilon}{2}\ln\frac{4\pi\mu^2}{\Delta} + \mathcal{O}(\epsilon^2)\right) \\
&= \frac{2}{\epsilon} + \ln\frac{4\pi\mu^2}{\Delta} - \gamma + \mathcal{O}(\epsilon) = \frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{\Delta} + \mathcal{O}(\epsilon),
\end{aligned}$$

where $\tilde{\mu}^2 := 4\pi\mu^2 e^{-\gamma}$. Thus, our final result is

$$\Pi^{(2)}(q^2) = -\underbrace{\frac{2\cdot 4g^2}{(4\pi)^2}}_{=2\alpha/\pi}\int_0^1 dx\, x(1-x)\left(\frac{2}{\epsilon}+\ln\frac{\tilde{\mu}^2}{\Delta}\right),$$

where $\quad \Delta = -x(1-x)q^2 + m^2.$

Note, that we give $\Pi(q^2)$ at this point, whereas the quantity appearing in the formula above is $i\Pi(q^2)$. We have also dropped all terms vanishing by $\epsilon \to 0$.

13.4.7 Adding up 1PIs

We can write the sum of all 1PI's,

$$\mathrm{FT}\langle\Omega|\mathcal{T}\, A^\mu(x)A^\nu(y)|\Omega\rangle =$$

using Feynman rules as

$$\begin{aligned}
\mathrm{FT}\,\langle\Omega|\mathcal{T}\, A_\mu(x)A_\nu(y)|\Omega\rangle &= \left(\frac{-i\eta_{\mu\nu}}{q^2}\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ i\Pi^{\rho\sigma}(q)\ \ \frac{-i\eta_{\sigma\nu}}{q^2}\right)\\
&\quad+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ i\Pi^{\rho\sigma}(q)\ \ \frac{-i\eta_{\sigma\kappa}}{q^2}\ \ i\Pi^{\kappa\eta}(q)\ \ \frac{-i\eta_{\eta\nu}}{q^2}\right)+\cdots\\
&=\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ i(q^2\eta^{\rho\sigma}-q^\rho q^\sigma)\Pi(q^2)\ \ \frac{-i\eta_{\sigma\nu}}{q^2}\right)\\
&\quad+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ i(q^2\eta^{\rho\sigma}-q^\rho q^\sigma)\Pi(q^2)\ \ \frac{-i\eta_{\sigma\kappa}}{q^2}\ \ i(q^2\eta^{\kappa\eta}-q^\kappa q^\eta)\Pi(q^2)\ \ \frac{-i\eta_{\eta\nu}}{q^2}\right)+\cdots\\
&=\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ \left(\eta^\rho_\nu-\frac{q^\rho q_\nu}{q^2}\right)\Pi(q^2)\right)\\
&\quad+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ \left(\eta^\rho_\kappa-\frac{q^\rho q_\kappa}{q^2}\right)\Pi(q^2)\ \ \left(\eta^\kappa_\nu-\frac{q^\kappa q_\nu}{q^2}\right)\Pi(q^2)\right)+\cdots\\
&=\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ \Delta^\rho_\nu\Pi(q^2)\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ \Delta^\rho_\kappa\Pi(q^2)\ \Delta^\kappa_\nu\Pi(q^2)\right)+\cdots,
\end{aligned}$$

where

$$\Delta^\mu_\nu \coloneqq \eta^\mu_\nu - q^\mu q_\nu/q^2$$

$$\Longrightarrow \qquad \Delta^\mu_\sigma\Delta^\sigma_\nu = \left(\eta^\mu_\sigma-\frac{q^\mu q_\sigma}{q^2}\right)\left(\eta^\sigma_\nu-\frac{q^\sigma q_\nu}{q^2}\right)=\eta^\mu_\nu-\frac{q^\mu q_\nu}{q^2}-\frac{q^\mu q_\nu}{q^2}+\frac{q^\mu q_\sigma}{q^2}\frac{q^\sigma q_\nu}{q^2}=\Delta^\mu_\nu.$$

Using this last identity, we find

$$\begin{aligned}
\mathrm{FT}\,\langle\Omega|\mathcal{T}\, A_\mu(x)A_\nu(y)|\Omega\rangle &= \left(\frac{-i\eta_{\mu\nu}}{q^2}\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ \Delta^\rho_\nu\Pi(q^2)\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ \Delta^\rho_\nu\Pi^2(q^2)\right)+\cdots\\
&=\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\right)\left(\Delta^\rho_\nu\sum_{n=1}^\infty\Pi^n(q^2)\right)\\
&=\frac{-i\eta_{\mu\nu}}{q^2}+\frac{-i\eta_{\mu\rho}}{q^2}\left(\eta^\rho_\nu-\frac{q^\rho q_\nu}{q^2}\right)\left(\frac{1}{1-\Pi(q^2)}-1\right)\\
&=\frac{-i\eta_{\mu\nu}}{q^2}+\frac{-i}{q^2}\left(\eta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{1-\Pi(q^2)}-\frac{-i}{q^2}\left(\eta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\\
&=\frac{-i}{q^2}\left(\eta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{1-\Pi(q^2)}+\frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}.
\end{aligned}$$

If we calculate a S-matrix element, this full photon propagator will be connected at one of its two ends to the rest of the Feynman diagram. This rest of the Feynman diagram can be described by an amplitude $\mathcal{M}^\nu$, which carries an index ν, since it must be connected to the polarization vector ε_ν of the external photon, which is not included in $\mathcal{M}^\nu$. Thus, the full propagator and the rest of the diagram together form the amplitude

$$\left(\mathrm{FT}\,\langle\Omega|\mathcal{T}\,A_\mu(x)A_\nu(y)|\Omega\rangle\right)\mathcal{M}^\nu.$$

When we now plug in the formula, we just derived for the full photon propagator, the momenta q_ν meet the amplitude $\mathcal{M}^\nu$ and vanish by the Ward identity. Thus, we only need to consider

$$\mathrm{FT}\,\langle\Omega|\mathcal{T}\,A_\mu(x)A_\nu(y)|\Omega\rangle = \frac{-i\eta_{\mu\nu}}{q^2}\frac{1}{1-\Pi(q^2)}.$$

13.5 Cancellation of UV Divergences

13.5.1 Electron Propagator with Renormalized Sigma

Using

$$Z_2 = 1+\delta_2, \qquad m_0 = m - \Delta m$$

from section 13.3, we find

$$\begin{aligned}
\frac{1}{Z_2}\frac{i}{\not p - m_0 - \Sigma(\not p)} &= \frac{1}{1+\delta_2}\frac{i}{\not p - m + \Delta m - \Sigma(\not p)}\\
&= \frac{i}{\not p - m + \Delta m - \Sigma(\not p) + \delta_2\left(\not p - m + \Delta m - \Sigma(\not p)\right)} = \frac{i}{\not p - m - \Sigma_R(\not p)}\\
\Leftrightarrow \qquad \frac{i}{\not p - m_0 - \Sigma(\not p)} &= \frac{iZ_2}{\not p - m - \Sigma_R(\not p)},
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_R(\not p) &= \Sigma(\not p) - \Delta m - \delta_2\left(\not p - m + \Delta m - \Sigma(\not p)\right)\\
&= \Sigma^{(2)}(\not p) - \Delta m^{(2)} - \delta_2^{(2)}(\not p - m) + \mathcal{O}(\alpha^2).
\end{aligned}$$

13.5.2 Renormalized Sigma is UV Finite

Plugging in the explicit expressions, we find that $\Sigma_R^{(2)}$ (the order-α contribution to Σ_R) is – except for the IR limit $\nu \to 0$ – indeed finite; that is, the UV divergences cancel:

$$\begin{aligned}
\Sigma_R^{(2)}(\not p) &= \frac{\alpha}{2\pi}\int_0^1 dx\,\Bigg((2m - x\not p)\ln\frac{x\Lambda^2}{\Delta} - (2-x)m\ln\frac{x\Lambda^2}{\Delta^0}\\
&\qquad\qquad - \left(\frac{2(2-x)x(1-x)m^2}{\Delta^0} - x\ln\frac{x\Lambda^2}{\Delta^0}\right)(\not p - m)\Bigg) + \mathcal{O}(\alpha^2)\\
&= \frac{\alpha}{2\pi}\int_0^1 dx\,\Bigg((2m - x\not p)\ln\frac{x\Lambda^2}{\Delta} - (2m - xm)\ln\frac{x\Lambda^2}{\Delta^0}\\
&\qquad\qquad - \frac{2(2-x)x(1-x)m^2}{\Delta^0}(\not p - m) + (x\not p - xm)\ln\frac{x\Lambda^2}{\Delta^0}\Bigg) + \mathcal{O}(\alpha^2)\\
&= \frac{\alpha}{2\pi}\int_0^1 dx\,\Bigg(2m\left(\ln\frac{x\Lambda^2}{\Delta} - \ln\frac{x\Lambda^2}{\Delta^0}\right) - x\not p\left(\ln\frac{x\Lambda^2}{\Delta} - \ln\frac{x\Lambda^2}{\Delta^0}\right)\\
&\qquad\qquad - \frac{2(2-x)x(1-x)m^2}{\Delta^0}(\not p - m)\Bigg) + \mathcal{O}(\alpha^2)
\end{aligned}$$

$$= \frac{\alpha}{2\pi}\int_0^1 dx \left((2m - x\not{p}) \ln\frac{\Delta^0}{\Delta} - (\not{p} - m)\frac{\widetilde{\Delta}}{\Delta^0} \right) + \mathcal{O}(\alpha^2),$$

where

$$\Delta = (1-x)m^2 - x(1-x)p^2 + x\nu^2, \qquad \Delta^0 = \Delta(p^2 = 0) = (1-x)^2 m^2 + x\nu^2,$$

$$\widetilde{\Delta} = 2(2-x)x(1-x)m^2.$$

Note that for $\not{p} = m$, the second term of $\Sigma_R(\not{p})$ obviously vanishes and also the first term gives zero since

$$\ln\frac{\Delta^0}{\Delta}\bigg|_{\not{p}=m} = \ln\frac{(1-x)^2m^2 + x\nu^2}{(1-x)m^2 - x(1-x)m^2 + x\nu^2} = \ln\frac{(1-x)^2m^2 + x\nu^2}{(1-x)^2m^2 + x\nu^2} = 0;$$

hence $\Sigma_R(\not{p} = m) = 0$.

13.5.3 Photon Propagator with Renormalized Pi

Using

$$Z_3 = 1 + \delta_3$$

from section 13.4, we find

$$\frac{1}{Z_3}\frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1 - \Pi(q^2)} = \frac{1}{1+\delta_3}\frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1 - \Pi(q^2)} = \frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1 - \Pi(q^2) + \delta_3 - \delta_3\Pi(q^2)}$$
$$= \frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1 - \Pi_R(q^2)}$$

$$\Leftrightarrow \qquad \frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1 - \Pi(q^2)} = \frac{-i\eta^{\mu\nu}}{q^2}\frac{Z_3}{1 - \Pi_R(q^2)},$$

where

$$\Pi_R(q^2) = \Pi(q^2) - \delta_3 + \delta_3\Pi(q^2) = \Pi^{(2)}(q^2) - \delta_3^{(2)} + \mathcal{O}(\alpha^2).$$

13.5.4 Renormalized Pi is UV Finite

Plugging in the explicit expressions, we find that Π_R is indeed finite; that is, the UV divergences cancel:

$$\Pi_R^{(2)}(q^2) = \Pi^{(2)}(q^2) - \delta_3^{(2)} = -\frac{2\alpha}{\pi}\int_0^1 dx \left(x(1-x)\left(\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{\Delta}\right) - x(1-x)\left(\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{m^2}\right)\right)$$
$$= -\frac{2\alpha}{\pi}\int_0^1 dx \left(x(1-x)\ln\frac{m^2}{\Delta}\right),$$

where

$$\Delta = -x(1-x)q^2 + m^2.$$

Obviously, $\Pi_R^{(2)}(0) = 0$.

13.5.5 Vertex Factor with Renormalized Gamma

Using $Z_1 = 1 + \delta_1 = 1 + \delta_1^{(2)} + \mathcal{O}(\alpha^2)$ as well as $F_2(q^2) = 0 + \mathcal{O}(\alpha^2)$, we find

$$
\begin{aligned}
ig_0 Z_1 \Gamma^\mu &= ig_0 Z_1 \left(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right) \\
&= ig_0 \left(\gamma^\mu \left(1 + \delta_1^{(2)}\right) \left(1 + \delta F_1(q^2)\right) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) + \mathcal{O}(\alpha^2) \right) \\
&= ig_0 \left(\gamma^\mu \left(1 + \delta F_1(q^2) + \delta_1^{(2)}\right) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) + \mathcal{O}(\alpha^2) \right) \\
&= ig_0 \left(\gamma^\mu F_{1R}^{(2)}(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) + \mathcal{O}(\alpha^2) \right)
\end{aligned}
$$

$$
\Leftrightarrow \qquad ig_0 \Gamma^\mu = \frac{ig}{Z_1 \sqrt{Z_3}} \left(\gamma^\mu F_{1R}^{(2)}(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) + \mathcal{O}(\alpha^2) \right),
$$

where

$$
F_{1R}^{(2)} = 1 + \delta F_1(q^2) + \delta_1^{(2)}, \qquad g := \sqrt{Z_3} g_0.
$$

13.5.6 The Renormalized Gamma is Finite

The only potentially divergent quantity within Γ_R is F_{1R}. However, plugging in the explicit expressions, we find that F_{1R} is indeed finite; that is, the UV divergences cancel:

$$
\begin{aligned}
F_{1R}(q^2) &= 1 + \delta F_1(q^2) + \underbrace{\delta_1^{(2)}}_{=-\delta F_1(0)} \\
&= 1 + \frac{\alpha}{2\pi} \int D\vec{x} \left(\ln \frac{x\Lambda^2}{\Delta} + \frac{(1-y)(1-z)q^2 + (1-4x+x^2)m^2}{\Delta} \right) \\
&\quad - \frac{\alpha}{2\pi} \int D\vec{x} \left(\ln \frac{x\Lambda^2}{\Delta^0} + \frac{(1-4x+x^2)m^2}{\Delta^0} \right) \\
&= \frac{\alpha}{2\pi} \int D\vec{x} \left(\ln \frac{\Delta^0}{\Delta} + \frac{(1-y)(1-z)q^2 + (1-4x+x^2)m^2}{\Delta} - \frac{(1-4x+x^2)m^2}{\Delta^0} \right).
\end{aligned}
$$

Note, that all UV divergences (in form of the regulator Λ) have cancelled. Here,

$$
\Delta = -q^2 xy + (1-x)^2 m^2 + x\nu^2, \qquad \Delta^0 = (1-x)^2 m^2 + x\nu^2.
$$

13.5.7 Absorbing the Field Strength Renormalizations into the Vertex Factors

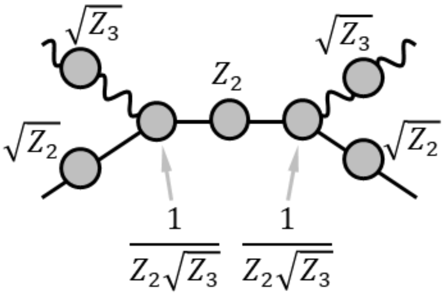
Each full electron propagator can be written as a finite renormalized propagator times a factor Z_2. Each photon propagator can be written as a finite renormalized propagator times a factor Z_3. And each full vertex factor can be written as a finite renormalized vertex factor times a factor $1/Z_2\sqrt{Z_3}$ (if $Z_1 = Z_2$)

$$
\begin{aligned}
&\frac{i}{\not p - m_0 - \Sigma(\not p)} = \frac{i}{\not p - m - \Sigma_R(\not p)} \cdot Z_2, \\
&\frac{-i\eta^{\mu\nu}}{q^2} \frac{1}{1 - \Pi(q^2)} = \frac{-i\eta^{\mu\nu}}{q^2} \frac{1}{1 - \Pi_R(q^2)} \cdot Z_3, \\
&ig_0 \Gamma^\mu(q) = ig\Gamma_R^\mu \cdot \frac{1}{Z_2\sqrt{Z_3}} \qquad \text{if} \qquad Z_1 = Z_2.
\end{aligned}
$$

Note, that on the right-hand side, only the physical mass and charge $m, g = e$ appear.

External particles, that do not enter via propagators into the diagram, come – according to the Feynman rules in section 8.2 – with factor $\sqrt{Z_2}, \sqrt{Z_3}$ respectively.

Each internal propagator is connected to two vertices; we can therefore split its factor Z_i into two factors $\sqrt{Z_i}\sqrt{Z_i}$ and move each of them to one of the adjacent vertices. In the end, all factors Z_i will cancel precisely. For example, for Compton scattering, the factors Z_i are distributed as follows:



We have just the right number of Z_i in the numerator and denominator to cancel.

Why are the external legs only contribute a factor $\sqrt{Z_i}$, despite having a full propagator blob? Or in other words: Why do external legs have a propagator blob, despite we do not write down propagators for external lines according to Feynman rules? To understand this, we need to consider the "alternative form" of the LSZ reduction formula of section 7.5. It connects the S-matrix element (the thing that appears in the cross section) with the corresponding n-point function as follows

$$\left(\Pi_{i=1}^{n}\frac{i\sqrt{Z}}{q_i^2-m^2}\right)\left(\Pi_{j=1}^{m}\frac{i\sqrt{Z}}{p_j^2-m^2}\right)S_{\beta\alpha} = \mathrm{FT}\,\langle\Omega|\mathcal{T}\,\phi(x_1)\cdots\phi(y_1)\cdots|\Omega\rangle$$

$$\Leftrightarrow \qquad S_{\beta\alpha} = \left(\Pi_{i=1}^{n}\frac{q_i^2-m^2}{i\sqrt{Z}}\right)\left(\Pi_{i=1}^{n}\frac{p_j^2-m^2}{i\sqrt{Z}}\right)\mathrm{FT}\,\langle\Omega|\mathcal{T}\,\phi(x_1)\cdots\phi(y_1)\cdots|\Omega\rangle.$$

For QED, the products obviously contain the fermion and photon propagators with their field strength renormalization factors Z_i. Note that only for external particles there is a factor $\left(q_i^2-m^2\right)/i\sqrt{Z}$. That means: The n-point functions do contain (full) propagators like

$$\frac{iZ_2}{\not{p}-m-\Sigma_R(\not{p})} \qquad \xrightarrow[\not{p}\to m]{\text{external particle}} \qquad \frac{iZ_2}{\not{p}-m},$$

also for *external* particles. However, external particles are on-shell and obey $\not{p}=m$ and hence $\Sigma_R(m)=0$. For cross sections, we do not need the n-point function, but the S-matrix element and for the S-matrix element the propagators of external particles are cancelled by the LSZ reduction formula above. However, the LSZ reduction cancels only a factor of $\sqrt{Z_i}$ and hence a factor of $Z_i/\sqrt{Z_i}=\sqrt{Z_i}$ remains.

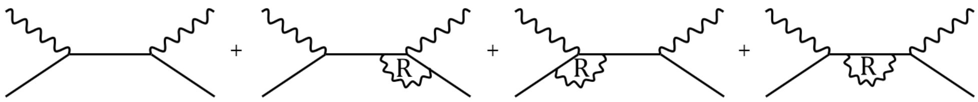
Obviously, this cancellation of the Z_i's works to all orders in perturbation theory (if $Z_1=Z_2$). The amplitude that corresponds to the diagram above – including all self-energies and the full vertex correction – reads, according to Feynman rules

$$\begin{aligned}&\left(\sqrt{Z_2}\bar{u}\right)\left(\sqrt{Z_3}\varepsilon_\mu\right)\left(\frac{ig}{Z_2\sqrt{Z_3}}\Gamma_R^\mu\right)\frac{iZ_2}{\not{p}-m-\Sigma_R(\not{p})}\left(\frac{ig}{Z_2\sqrt{Z_3}}\Gamma_R^\nu\right)\left(\sqrt{Z_2}u\right)\left(\sqrt{Z_3}\varepsilon_\nu\right)\\ &\quad = -4\pi\alpha\,\bar{u}\varepsilon_\mu\,\Gamma^\mu\frac{i}{\not{p}-m-\Sigma_R(\not{p})}\Gamma^\nu\,u\varepsilon_\nu.\end{aligned}$$

If we are interested in into the NLO correction only, we simply expand this amplitude in α:

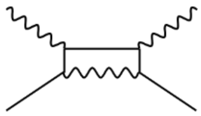
$$
\begin{aligned}
&-4\pi\alpha\, \bar{u}\varepsilon_\mu\, \Gamma_R^\mu \frac{i}{\not{p} - m - \Sigma_R(\not{p})} \Gamma_R^\nu\, u\varepsilon_\nu \\
&\quad = -4\pi\alpha\, \bar{u}\varepsilon_\mu \left(\gamma^\mu + \delta\Gamma_R^\mu\right) \left(\frac{i}{\not{p} - m} + \frac{i}{\not{p} - m}\left(-i\Sigma_R(\not{p})\right)\frac{i}{\not{p} - m}\right) \left(\gamma^\nu + \delta\Gamma_R^\nu\right) u\varepsilon_\nu + \mathcal{O}(\alpha^3) \\
&\quad = -4\pi\alpha\, \bar{u}\varepsilon_\mu \Bigg(\left(\gamma^\mu \frac{i}{\not{p} - m}\gamma^\nu\right) + \left(\gamma^\mu \frac{i}{\not{p} - m}\delta\Gamma_R^\nu\right) + \left(\delta\Gamma_R^\mu \frac{i}{\not{p} - m}\gamma^\nu\right) \\
&\qquad\qquad + \left(\gamma^\mu \frac{i}{\not{p} - m}\left(-i\Sigma_R(\not{p})\right)\frac{i}{\not{p} - m}\gamma^\nu\right)\Bigg) u\varepsilon_\nu + \mathcal{O}(\alpha^3).
\end{aligned}
$$

Those four terms correspond to the diagrams



The letter "R" denotes that those loop corrections are described by the renormalized (finite) Γ_R^μ, Σ_R instead of the infinite Γ^μ, Σ.

Note that the full propagators of external (on-shell) particles are cancelled in the S-matrix element due to LSZ reduction, hence corrections at the external legs do not contribute. However, one correction of order α^2 *is* missing, namely the following one:



However, this diagram is UV finite: The denominator receives one factor of momentum from each electron propagator and two from the photon propagator, makes seven in total. The loop integral d^4p contributes only four factors of momentum. Hence, the loop integral will be UV convergent and we do not need to show any cancellations in this case.

13.5.8 Equality of the Vertex and Fermion Renormalization Factor

The equality $Z_1 = Z_2$ to order-α means

$$
Z_1 = Z_2 \qquad \Leftrightarrow \qquad 1 + \delta_1^{(2)} + \mathcal{O}(\alpha^2) = 1 + \delta_2^{(2)} + \mathcal{O}(\alpha^2) \qquad \Leftrightarrow \qquad \delta_1^{(2)} = \delta_2^{(2)}.
$$

In the sections 13.2 and 13.3 we have derived the following expressions:

$$
\begin{aligned}
\delta_1^{(2)} &= -\frac{\alpha}{2\pi}\int_0^1 dx\,(1-x)\left(\ln\frac{x\Lambda^2}{(1-x)^2m^2 + xv^2} + \frac{(1-4x+x^2)m^2}{(1-x)^2m^2 + xv^2}\right), \\
\delta_2^{(2)} &= \frac{\alpha}{2\pi}\int_0^1 dx\left(-x\ln\frac{x\Lambda^2}{(1-x)^2m^2 + xv^2} + 2(2-x)\frac{x(1-x)m^2}{(1-x)^2m^2 + xv^2}\right).
\end{aligned}
$$

We are now going to proof, that those expressions are equal by showing that the difference vanishes:

$$
\begin{aligned}
\delta_2^{(2)} - \delta_1^{(2)} = \frac{\alpha}{2\pi}\int_0^1 dx\Bigg((1-2x)\ln\frac{x\Lambda^2}{(1-x)^2m^2 + xv^2} &+ 2(2-x)\frac{x(1-x)m^2}{(1-x)^2m^2 + xv^2} \\
&+ (1-x)\frac{(1-4x+x^2)m^2}{(1-x)^2m^2 + xv^2}\Bigg).
\end{aligned}
$$

Considering the first term only, let's perform integration by parts:

$$\int_0^1 dx\,(1-2x)\ln\frac{x\Lambda^2}{(1-x)^2m^2+x\nu^2}$$
$$= -\int_0^1 dx\,x(1-x)\frac{(1-x)^2m^2+x\nu^2}{x\Lambda^2}\left(\frac{\Lambda^2}{(1-x)^2m^2+x\nu^2}\right.$$
$$\left.-\frac{x\Lambda^2}{((1-x)^2m^2+x\nu^2)^2}\left(-2(1-x)m^2+\nu^2\right)\right)$$
$$= -\int_0^1 dx\,x(1-x)\left(\frac{1}{x}-\frac{-2(1-x)m^2+\nu^2}{(1-x)^2m^2+x\nu^2}\right)$$
$$= -\int_0^1 dx\,x(1-x)\left(\frac{(1-x)^2m^2+x\nu^2}{x((1-x)^2m^2+x\nu^2)}-\frac{-2x(1-x)m^2+x\nu^2}{x((1-x)^2m^2+x\nu^2)}\right)$$
$$= -\int_0^1 dx\,x(1-x)\left(\frac{(1+x)(1-x)m^2}{x((1-x)^2m^2+x\nu^2)}\right)=\int_0^1 dx\,\frac{-(1+x)(1-x)^2m^2}{(1-x)^2m^2+x\nu^2}.$$

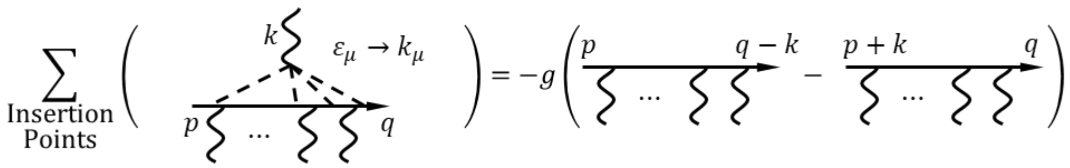
Plugging this result in, we find

$$\delta_2^{(2)}-\delta_1^{(2)}$$
$$=\frac{\alpha}{2\pi}\int_0^1 dx\left(\frac{-(1+x)(1-x)^2m^2}{(1-x)^2m^2+x\nu^2}+\frac{2(2-x)x(1-x)m^2}{(1-x)^2m^2+x\nu^2}\right.$$
$$\left.+\frac{(1-x)(1-4x+x^2)m^2}{(1-x)^2m^2+x\nu^2}\right)$$
$$=\frac{\alpha}{2\pi}\int_0^1 dx\,\frac{(1-x)m^2}{(1-x)^2m^2+x\nu^2}\left(-(1+x)(1-x)+2(2-x)x+(1-4x+x^2)\right)$$
$$=\frac{\alpha}{2\pi}\int_0^1 dx\,\frac{(1-x)m^2}{(1-x)^2m^2+x\nu^2}\underbrace{\left((-1+x^2)+(4x-2x^2)+(1-4x+x^2)\right)}_{=0}=0$$

$$\Leftrightarrow \qquad \delta_2^{(2)}=\delta_1^{(2)}.$$

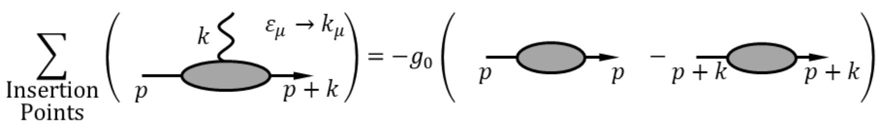
13.5.9 Proof of the Equality of Z1 and Z2 to all Orders in Perturbation Theory

Recall the Ward-Takahashi identity from section 11.5, which we gave pictorially as



(where g is actually the bare charge, which we by now call g_0). The straight line is an arbitrary electron line through a Feynman diagram, connecting to external electrons. The photon lines attached to it *below* can be either external or internal, the latter case means they are connected to another (or the same) electron line.

Let us apply this general formula to the special case of a diagram with only to external electrons and the photon with momentum k shall be the only external photon. Then, the general picture of above looks as follows:



Despite having only three external particles, their interaction can be arbitrarily complicated. The sum of all possible diagrams with two electrons photons and one external photon is described by the blob on the left-hand side. That is just the vertex correction to all orders in perturbation theory. On the right-hand side, we have simply the exact electron propagator for different momenta. This is basically the *simplest* special case of the Ward-Takahashi identity.

Let us call the full photon propagator with momentum $S(p)$. Then, we know from section 13.3 and 13.5 that

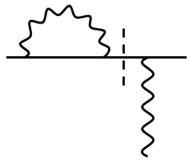
$$S(p) := \frac{i}{\not{p} - m_0 - \Sigma(\not{p})} = \frac{iZ_2}{\not{p} - m - \Sigma_R(\not{p})}.$$

Then, the right-hand side of the picture above is simply $-g_0\big(S(p) - S(p+k)\big)$.

On the other hand, we wrote the full vertex correction in 13.2 as

$$ig_0\Gamma^\mu(k),$$

where Γ^μ also includes the sum over the insertion points. The Γ^μ is contracted with the polarization vector of the external photon, which will become $\varepsilon_\mu \to k_\mu$ for the Ward-Takahashi identity. However, Γ^μ did not include reducible diagrams like



However, the blob on the left-hand side above does include such diagrams. Obviously, such a loop *next to* a vertex is simply a correction contained in an electron propagator. Thus, the left-hand side of the pictorial equation above is $S(p)\big(ig_0k_\mu\Gamma^\mu\big)S(p+k)$. Hence, we find

$$S(p)\left(ig_0k_\mu\Gamma^\mu(k)\right)S(p+k) = -g_0\big(S(p) - S(p+k)\big)$$

$$\Leftrightarrow \qquad ik_\mu\Gamma^\mu(k) = -\big(S^{-1}(p+k) - S^{-1}(p)\big)$$

$$\Leftrightarrow \qquad ik_\mu\Gamma^\mu(k) = -\frac{1}{iZ_2}\not{k}.$$

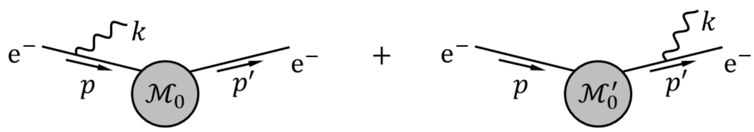
In the limit $k \to 0$, we have defined $\gamma^\mu = Z_1\,\Gamma^\mu(k \to 0)$ in section 13.2. For the equation above to hold for arbitrarily small values of k, we need to have

$$ik_\mu\gamma^\mu Z_1^{-1} = -\frac{1}{iZ_2}\not{k} \qquad \Leftrightarrow \qquad Z_1 = Z_2.$$

13.6 Soft Bremsstrahlung

13.6.1 Amplitude for One Soft Photon

Consider an electron, which undergoes an arbitrary scattering process with amplitude $\mathcal{M}_0$, and in addition emits one soft photon of a small momentum k either before or after the process:



Using the Feynman rules for QED, the amplitude of the whole process reads

$$
\begin{aligned}
i\mathcal{M} &= \bar{u}_{p'} \;\; \mathcal{M}_0 \;\; \frac{i}{\not{p} - \not{k} - m + i\epsilon} \;\; ig\gamma^\mu \;\; \varepsilon_{k\mu} u_p \;\; + \;\; \bar{u}_{p'} \varepsilon_{k\mu} \;\; ig\gamma^\mu \;\; \frac{i}{\not{p}' + \not{k} - m + i\epsilon} \mathcal{M}_0' \;\; u_p \\
&= ig\bar{u}_{p'} \left(\mathcal{M}_0 \frac{i}{\not{p} - \not{k} - m + i\epsilon} \not{\varepsilon}_k + \not{\varepsilon}_k \frac{i}{\not{p}' + \not{k} - m + i\epsilon} \mathcal{M}_0' \right) u_p.
\end{aligned}
$$

Note that, of course, $\mathcal{M}_0$ is a function of its incoming momenta, however, the correction of the small momentum $k \ll p, p'$ of the soft photon can be neglected:

$$
\mathcal{M}_0 \equiv \mathcal{M}_0(p', p - k) \approx \mathcal{M}_0(p', p), \qquad \mathcal{M}_0' \equiv \mathcal{M}_0(p' + k, p) \approx \mathcal{M}_0(p', p) = \mathcal{M}_0.
$$

Using some Dirac algebra, the amplitude can be further simplified. Those steps where already conducted in (>10.4.1); for more details see there.

First, we bring the γ-matrices to the denominator:

$$
i\mathcal{M} \approx -g\bar{u}_{p'} \left(\mathcal{M}_0 \frac{\not{p} - \not{k} + m}{(p - k)^2 - m^2 + i\epsilon} \not{\varepsilon}_k + \not{\varepsilon}_k \frac{\not{p}' + \not{k} + m}{(p' + k)^2 - m^2 + i\epsilon} \mathcal{M}_0 \right) u_p.
$$

There, in the denominator, the photon momenta k can be neglected compared to the electron momenta p, that is $\not{p} \pm \not{k} = \not{p}$. Next, we use $\not{a}\gamma^\nu = -\gamma^\nu \not{a} + 2a^\nu$ and $\gamma^\nu \not{a} = -\not{a}\gamma^\nu + 2a^\nu$ to get

$$
(\not{p} + m)\not{\varepsilon}_k u_p = (-\not{\varepsilon}_k \not{p} + 2p \cdot \varepsilon_k + m\not{\varepsilon}_k) u_p = \big(2p \cdot \varepsilon_k + \not{\varepsilon}_k(-\not{p} + m)\big) u_p = 2p \cdot \varepsilon_k \, u_p,
$$

$$
\bar{u}_{p'} \not{\varepsilon}_k (\not{p}' + m) = \bar{u}_{p'}(-\not{p}' \not{\varepsilon}_k + 2p' \cdot \varepsilon_k + \not{\varepsilon}_k m) = \bar{u}_{p'}\big((-\not{p}' + m)\not{\varepsilon}_k + 2p' \cdot \varepsilon_k\big) = \bar{u}_{p'} \, 2p' \cdot \varepsilon_k.
$$

In the denominator we find

$$
(p - k)^2 - m^2 = -2p \cdot k, \qquad (p' + k)^2 - m^2 = 2p' \cdot k.
$$

Thus, the amplitude becomes in the limit $\epsilon \to 0$

$$
i\mathcal{M} \approx -g\bar{u}_{p'} \left(\mathcal{M}_0 \frac{2p \cdot \varepsilon_k}{-2p \cdot k} + \frac{2p' \cdot \varepsilon_k}{2p' \cdot k} \mathcal{M}_0 \right) u_p = -g \left(\frac{p' \cdot \varepsilon_k}{p' \cdot k} - \frac{p \cdot \varepsilon_k}{p \cdot k} \right) \bar{u}_{p'} \mathcal{M}_0 u_p.
$$

13.6.2 Evaluating the Photon Momentum Integral

BRINGING THE INTEGRAL INTO A NICE SHAPE:

The next step is obviously to evaluate the integral over the soft photon momentum k,

$$
\begin{aligned}
\mathcal{I} &:= \int_\nu^L d\tilde{k} \sum_{\lambda=1,2} \left| \frac{p' \cdot \varepsilon_k}{p' \cdot k} - \frac{p \cdot \varepsilon_k}{p \cdot k} \right|^2 = \int_\nu^L d\tilde{k} \sum_{\lambda=1,2} \left(\varepsilon_{k\mu} \left(\frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right) \right)^2 \\
&= \int_\nu^L d\tilde{k} \sum_{\lambda=1,2} \varepsilon_{k\mu} \varepsilon_{k\nu} \left(\frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right) \left(\frac{p'^\nu}{p' \cdot k} - \frac{p^\nu}{p \cdot k} \right).
\end{aligned}
$$

We the sum over polarizations from (>6.3.1). To see that we can simply use $\sum_\lambda \varepsilon_{k\mu}\varepsilon_{k\nu} = -\eta_{\mu\nu}$, we can use the standard trick and replace a $\epsilon_{k\mu}$ by a k_μ:

$$
k_\mu \varepsilon_{k\nu} \left(\frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right) \left(\frac{p'^\nu}{p' \cdot k} - \frac{p^\nu}{p \cdot k} \right) = k_\mu \varepsilon_{k\nu} \left(\frac{p' \cdot k}{p' \cdot k} - \frac{p \cdot k}{p \cdot k} \right) \left(\frac{p'^\nu}{p' \cdot k} - \frac{p^\nu}{p \cdot k} \right) = 0
$$

(this is just an explicit proof that the Ward identity holds in this special case). Thus, we find

$$
\begin{aligned}
\mathcal{I} &= -\int_\nu^L d\tilde{k} \, \eta_{\mu\nu} \left(\frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right) \left(\frac{p'^\nu}{p' \cdot k} - \frac{p^\nu}{p \cdot k} \right) = -\int_\nu^L d\tilde{k} \, \left(\frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right) \left(\frac{p'_\mu}{p' \cdot k} - \frac{p_\mu}{p \cdot k} \right) \\
&= \int_\nu^L d\tilde{k} \, \left(\frac{2p \cdot p'}{(p \cdot k)(p' \cdot k)} - \frac{m^2}{(p' \cdot k)^2} - \frac{m^2}{(p \cdot k)^2} \right) =: \mathcal{I}_{pp'} - m^2\big(\mathcal{I}_{p'} + \mathcal{I}_p\big),
\end{aligned}
$$

where

$$\mathcal{I}_{pp'} := \int_\nu^L d\tilde{k}\, \frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)}, \qquad \mathcal{I}_p := \int_\nu^L d\tilde{k}\, \frac{1}{(p\cdot k)^2}$$

where we split up $\mathcal{I}$ into its parts.

COMPUTE THE INTEGRAL $\mathcal{I}_p$:

First, consider $\mathcal{I}_p$. If we write the scalar product as

$$p\cdot k = \omega_p\omega_k - \vec{p}\vec{k} = \omega_p\omega_k - \omega_k\sqrt{\omega_p^2 - m^2}\cos\theta = \omega_p\omega_k\big(1-\beta_p\cos\theta\big),$$

where $\beta_p := \sqrt{1-m^2/\omega_p^2}$ and $\omega_p^2 := m^2 + \vec{p}^2$, $\omega_k^2 := \vec{k}^2$. The integral can be evaluated as

$$\begin{aligned}
\mathcal{I}_p &= \int_\nu^L d\tilde{k}\frac{1}{(p\cdot k)^2} = \int_\nu^L \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{1}{\omega_p^2\omega_k^2\big(1-\beta_p\cos\theta\big)^2} \\
&= 2\pi\int_\nu^L \frac{\omega_k^2\, d\omega_k}{(2\pi)^3 2\omega_k}\int_{-1}^1 d\cos\theta\,\frac{1}{\omega_p^2\omega_k^2\big(1-\beta_p\cos\theta\big)^2} = 2\pi\frac{1}{\omega_p^2}\frac{2}{1-\beta_p^2}\int_\nu^L\frac{d\omega_k}{(2\pi)^3 2\omega_k} \\
&= \frac{1}{(2\pi)^2 m^2}\ln\frac{L}{\nu},
\end{aligned}$$

where we also simplified $1-\beta_p^2 = m^2/\omega_p^2$. Note, that $\mathcal{I}_p$ is IR divergent (limit $\nu\to 0$).

COMPUTE THE INTEGRAL $\mathcal{I}_{pp'}$:

Slightly more complicated is the evaluation of the integral $\mathcal{I}_{pp'}$, we even need Feynman parameters for it. Using the identity for $1/AB$ from section 12.2, we can write

$$\begin{aligned}
\mathcal{I}_{pp'} &:= \int_\nu^L d\tilde{k}\,\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)} = \int_0^1 dx\int_\nu^L d\tilde{k}\,\frac{2p\cdot p'}{\big(x(p\cdot k)+(1-x)(p'\cdot k)\big)^2} \\
&= 2(p\cdot p')\int_0^1 dx\,\int_\nu^L d\tilde{k}\,\frac{1}{(Q(x)\cdot k)^2} = 2(p\cdot p')\int_0^1 dx\,\mathcal{I}_{Q(x)},
\end{aligned}$$

where $Q(x) := xp' + (1-x)p$. Since Q does not contain the integration variable k, it plays exactly the role of the momentum p inside $\mathcal{I}_p$, which is why we could give the momentum integral as $\mathcal{I}_{Q(x)}$. Note that the mass m^2 in the result for $\mathcal{I}_p$ above is the mass of p, that is $m^2 = p^2$, or

$$\mathcal{I}_p = \frac{1}{(2\pi)^2p^2}\ln\frac{L}{\nu} \qquad\Longrightarrow\qquad \mathcal{I}_{Q(x)} = \frac{1}{(2\pi)^2Q^2(x)}\ln\frac{L}{\nu}.$$

Plugging this result into $\mathcal{I}_{pp'}$, we find

$$\mathcal{I}_{pp'} = \frac{2p\cdot p'}{(2\pi)^2}\ln\frac{L}{\nu}\int_0^1 dx\,\frac{1}{Q^2(x)} = \frac{2p\cdot p'}{(2\pi)^2}\ln\frac{L}{\nu}\int_0^1 dx\,\frac{1}{(xp'+(1-x)p)^2}$$

It now appears to be very easy: Just use the identity of the Feynman parameter again and the dx integral will be equal to $1/(p'\cdot p)$. Unfortunately however, the square in the denominator of our integral here is a four-momentum scalar product whereas the square in the Feynman parameter identity is a simple square of numbers. Thus, we must proceed differently. If we define

$$q^2 := (p-p')^2 = 2m^2 - 2p\cdot p',$$

we find that we can write

$$Q^2(x) = (xp'+(1-x)p)^2$$

$$
\begin{aligned}
&= \underbrace{x^2m^2 + (1-x)^2m^2}_{=x^2m^2+m^2-2xm^2+x^2m^2} + 2x(1-x)p\cdot p' \\
&= m^2 \underbrace{-\,2xm^2 + 2x^2m^2}_{=-2m^2x(1-x)} + 2x(1-x)p\cdot p' \\
&= m^2 + (-2m^2 + 2p\cdot p')x(1-x) \\
&= m^2 - q^2x(1-x).
\end{aligned}
$$

The complete integral now reads, using $2p\cdot p' = 2m^2 - q^2$,

$$
\mathcal{I}_{pp'} = \frac{1}{(2\pi)^2}\ln\frac{L}{\nu}\int_0^1 dx\ \frac{2m^2 - q^2}{m^2 - q^2x(1-x)}.
$$

COMBINING THE RESULTS:

Thus, the whole integral reads

$$
\begin{aligned}
\mathcal{I} = \mathcal{I}_{pp'} - m^2\big(\mathcal{I}_{p'} + \mathcal{I}_p\big) &= \frac{1}{(2\pi)^2}\ln\frac{L}{\nu}\int_0^1 dx\ \frac{2m^2 - q^2}{m^2 - q^2x(1-x)} - 2m^2\frac{1}{(2\pi)^2m^2}\ln\frac{L}{\nu} \\
&= \frac{1}{(2\pi)^2}\ln\frac{L}{\nu}\underbrace{\left(\int_0^1 dx\ \frac{2m^2 - q^2}{m^2 - q^2x(1-x)} - 2\right)}_{=:2f_{\mathrm{IR}}(q^2)} = \frac{1}{(2\pi)^2}\ln\frac{L^2}{\nu^2}f_{\mathrm{IR}}(q^2),
\end{aligned}
$$

where we put the factor 2 from $2f_{\mathrm{IR}}(q^2)$ as an exponent into the logarithm.

13.6.3 The Limit of Large Momentum

At least in the regime of $-q^2 \gg m^2$, we can also evaluate the integral $f_{\mathrm{IR}}(q^2)$: We can now neglect the squared masses, but then the integral over x does not converge anymore (note, that for a general $-q^2$, the integral is *not* divergent; this divergence is only because of our approximation). If we fix this issue by shifting the boundaries a little, we get

$$
\begin{aligned}
f_{\mathrm{IR}}(q^2) &\coloneqq \frac{1}{2}\left(\int_0^1 dx\frac{2m^2 - q^2}{m^2 - q^2x(1-x)} - 2\right) \overset{-q^2\gg m^2}{\approx} \frac{1}{2}\left(\int_{m^2/(-q^2)}^{1-m^2/(-q^2)} dx\frac{-q^2}{-q^2x(1-x)} - 2\right) \\
&= \frac{1}{2}\left(2\frac{-q^2}{-q^2}\ln\frac{-q^2}{m^2} - 2\right) \approx \ln\frac{-q^2}{m^2},
\end{aligned}
$$

where we used

$$
\int_a^b dx\frac{1}{x(1-x)} = 2(\operatorname{artanh}(1-2a) - \operatorname{artanh}(1-2b)) = \ln\frac{1-a}{a} - \ln\frac{1-b}{b} = \ln\frac{b(1-a)}{a(1-b)},
$$

which yields in the case $b = 1 - a$

$$
\int_a^{1-a} dx\frac{1}{x(1-x)} = \ln\frac{(1-a)(1-a)}{a\big(1-(1-a)\big)} = \ln\frac{(1-a)^2}{a^2} \overset{a\ll 1}{\approx} 2\ln a^{-1}.
$$

13.7 Infrared Divergence in the Vertex Factor

13.7.1 Infrared Divergence of the Renormalized Form Factor

Consider the renormalized form factor from section 13.5, where the expressions for δF_1 are given in section 13.2:

$$\begin{aligned}\delta F_{1R}(q^2) &:= \delta F_1(q^2) - \delta F_1(0)\\ &= \frac{\alpha}{2\pi}\int D\vec{x}\left(\ln\frac{(1-x)^2m^2 + xv^2}{-q^2yz + (1-x)^2m^2 + xv^2} + \frac{(1-y)(1-z)q^2 + (1-4x+x^2)m^2}{-q^2yz + (1-x)^2m^2 + xv^2}\right.\\ &\qquad\qquad\left. - \frac{(1-4x+x^2)m^2}{(1-x)^2m^2 + xv^2}\right).\end{aligned}$$

Here, we have already plugged in the expressions for Δ and Δ^0 and combined the two logarithms of $\delta F_1(q^2)$ and $\delta F_1(0)$, which cancelled the UV regulator Λ. Also, recall $D\vec{x} := dx\,dy\,dz\,\delta(1-x-y-z)$.

It can be shown, that the term with the logarithm is finite, also in the physical limit $v \to 0$. That is, it does not contribute to the IR divergence. Let us therefore drop this constant term (together with the 1 in front of the whole integral) for the following examination of the divergence.

In the limit $v \to 0$, the denominators blow up for $x \to 1$, which means, by the δ-function inside $D\vec{x}$, that $y, z \to 0$ (this also is a true statement for the denominator of the logarithm, but the logarithm makes the "blowing up" slow enough for the integral to converge). For the computation of the behaviour of F_{1R} at the IR divergence, it is sufficient to examine how the integral behaves in this region, where $x \to 1$ and $y, z \to 0$. Hence, let us set $x = 1$ and $y, z = 0$ in the numerators. Also, since we are interested into the limit $v \to 0$, the behaviour of the integral in this limit will not change when we replace xv^2 by v^2:

$$\delta F_{1R}(q^2) \approx \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z)\left(\frac{q^2 - 2m^2}{-q^2yz + (1-x)^2m^2 + v^2} - \frac{-2m^2}{(1-x)^2m^2 + v^2}\right).$$

Using

$$\int_0^1 dx\int_0^1 dy\int_0^1 dz\ f(x,y,z)\,\delta(1-x-y-z) = \int_0^1 dx\int_0^{1-x} dy\,f(x,y,1-x-y).$$

from (>13.2.8), we find

$$\delta F_{1R}(q^2) \approx \frac{\alpha}{2\pi}\int_0^1 dx\int_0^{1-x} dy\,\left(\frac{q^2 - 2m^2}{-q^2y(1-x-y) + (1-x)^2m^2 + v^2} - \frac{-2m^2}{(1-x)^2m^2 + v^2}\right).$$

We now substitute

$$w = 1-x, \qquad y = w\xi \qquad \Longrightarrow \qquad dx = -dw, \qquad dy = w\,d\xi,$$

where the last differential identity holds, since x or w are constants with respect to y and ξ. Thus, the integrals read

$$\int_0^1 dx\int_0^{1-x} dy = \int_{1-0}^{1-1}(-dw)\int_{0/w}^{(1-x)/w} w\,d\xi = -\int_1^0 \underbrace{w\,dw}_{=dw^2/2}\int_0^1 d\xi = \frac{1}{2}\int_0^1 dw^2\int_0^1 d\xi,$$

which yields when plugged in into the formula for $F_1(q^2)$

$$\begin{aligned}\delta F_{1R}(q^2) &\approx \frac{\alpha}{4\pi}\int_0^1 dw^2\int_0^1 d\xi\,\left(\frac{q^2 - 2m^2}{-q^2w\xi(w - w\xi) + m^2w^2 + v^2} - \frac{-2m^2}{m^2w^2 + v^2}\right)\\ &= \frac{\alpha}{4\pi}\int_0^1 d\xi\int_0^1 dw^2\,\left(\frac{q^2 - 2m^2}{(-q^2\xi(1-\xi) + m^2)w^2 + v^2} - \frac{-2m^2}{m^2w^2 + v^2}\right)\\ &= \frac{\alpha}{4\pi}\int_0^1 d\xi\,\left(\frac{q^2 - 2m^2}{-q^2\xi(1-\xi) + m^2}\ln\frac{-q^2\xi(1-\xi) + m^2 + v^2}{v^2} + 2\ln\frac{m^2 + v^2}{v^2}\right),\end{aligned}$$

where we used for $W := w^2$ the integral identity

$$\int_0^1 dW\,\frac{A}{BW + C} = \frac{A}{B}\ln\frac{B + C}{C}.$$

We can set $\nu^2 \to 0$ in the numerators. Furthermore, in the limit of $\nu \to 0$, the logarithms blow up no matter what the numerators are; anything proportional to q^2 or m^2 is effectively the same. We therefore write

$$\begin{aligned}\delta F_{1R}(q^2) &\approx \frac{\alpha}{4\pi}\int_0^1 d\xi\,\left(\frac{q^2-2m^2}{-q^2\xi(1-\xi)+m^2}\ln\frac{-q^2\text{ or }m^2}{\nu^2}+2\ln\frac{-q^2\text{ or }m^2}{\nu^2}\right)\\ &= -\frac{\alpha}{2\pi}\ln\frac{-q^2\text{ or }m^2}{\nu^2}\underbrace{\frac{1}{2}\int_0^1 d\xi\,\left(\frac{q^2-2m^2}{q^2\xi(1-\xi)-m^2}-2\right)}_{=f_{\mathrm{IR}}(q^2)}.\end{aligned}$$

Note, that it was exactly this function $f_{\mathrm{IR}}(q^2)$ that we also encountered in section 13.6.

13.8 Cancellation of Infrared Divergences

13.8.1 The Cross Section Including the Vertex Correction

If $d\sigma_0$ is the cross section that corresponds to the amplitude $\mathcal{M}_0$, that is

$$d\sigma_0 = c\,|\mathcal{M}_0|^2$$

with some proportionality c, then the cross section of the amplitude $\mathcal{M}_0(1+\delta F_{1R})$ reads

$$d\sigma_{\mathrm{V}} = c\,|\mathcal{M}_0(1+\delta F_{1R})|^2 = d\sigma_0\cdot(1+\delta F_{1R})^2 = d\sigma_0\cdot\big(1+2\delta F_{1R}+\mathcal{O}(\alpha^2)\big).$$

13.8.2 The Sum of the Cross Sections

Using the explicit results for δF_{1R} from section 13.7 and for $g^2\mathcal{J}$ from section 13.6, the total cross section reads

$$\begin{aligned}d\sigma &= d\sigma_{\mathrm{V}}+d\sigma_{\mathrm{B}} = d\sigma_0\cdot(1+2\delta F_{1R}+g^2\mathcal{J})\\ &= d\sigma_0\cdot\left(1-2\frac{\alpha}{2\pi}f_{\mathrm{IR}}(q^2)\ln\frac{-q^2\text{ or }m^2}{\nu^2}+\frac{\alpha}{\pi}f_{\mathrm{IR}}(q^2)\ln\frac{L^2}{\nu^2}+\text{IR finite}\right)\\ &= d\sigma_0\cdot\left(1-\frac{\alpha}{\pi}f_{\mathrm{IR}}(q^2)\ln\frac{-q^2\text{ or }m^2}{L^2}+\text{IR finite}\right)\end{aligned}$$

and is completely finite.

14.2 Imaginary Part of the Photon Self-Energy

14.2.1 Sign of the Mandelstam Variables

In a s-channel diagram, the two incoming particles with momenta p and k must have the same masses; therefore, the Mandelstam variable s reads

$$s = (p+k)^2 = p^2 + k^2 + 2p \cdot k = 2m^2 + 2\big(p_0 k_0 - \vec{p} \cdot \vec{k}\big).$$

Since

$$p_0 k_0 = \sqrt{\vec{p}^2 + m^2}\sqrt{\vec{k}^2 + m^2} > |\vec{p}||\vec{k}| \geq |\vec{p}||\vec{k}| \cos\theta = \vec{p} \cdot \vec{k},$$

s is obviously positive.

For t and u, it is not so simple to tell whether they are positive or negative. However, Mandelstam variables are Lorentz invariant and therefore we can switch into the centre of mass frame with the following momenta:

$$\begin{aligned} &\text{incoming particles:} \qquad && p = (E, |\vec{p}|\vec{e}_z), \qquad && k = (E, -|\vec{p}|\vec{e}_z), \\ &\text{outgoing particles:} \qquad && p' = (E, \vec{p}'), \qquad && k' = (E, -\vec{p}'). \end{aligned}$$

For simplicity, we also assumed in this case that all masses are equal. Since we are in the centre of mass frame, we have $|\vec{p}'| = |\vec{p}|$. For θ being the angle between $\vec{p}'$ and $\vec{e}_z$ (thus $\theta \in [0, \pi]$), we find

$$\begin{aligned} t &= (p-p')^2 = 2m^2 - 2p \cdot p' = 2m^2 - 2(E^2 - |\vec{p}|\vec{e}_z \cdot \vec{p}') = 2m^2 - 2(|\vec{p}|^2 + m^2 - |\vec{p}|^2 \cos\theta) \\ &= -2|\vec{p}|^2(1 - \cos\theta) < 0, \\ u &= (p-k')^2 = 2m^2 - 2p \cdot k' = 2m^2 - 2(E^2 + |\vec{p}|\vec{e}_z \cdot \vec{p}') = 2m^2 - 2(|\vec{p}|^2 + m^2 + |\vec{p}|^2 \cos\theta) \\ &= -2|\vec{p}|^2(1 + \cos\theta) < 0. \end{aligned}$$

Since q^2 just equals the Mandelstam variable of the considered channel, we have $q^2 < 0$ in the t and u channel and $q^2 > 0$ in the s channel.

14.2.2 Negative Argument of the Logarithm

The inequality

$$m^2 - x(1-x)q^2 < 0 \qquad \Leftrightarrow \qquad m^2 < x(1-x)q^2$$

must hold for all $x \in [0, 1]$, since this is the region of integration in $\hat{\Pi}(q^2)$. Within this region, $x(1-x)$ is at most $1/4$, thus we can write

$$m^2 < x(1-x)q^2 \leq \frac{1}{4}q^2 \qquad \Leftrightarrow \qquad q^2 > 4m^2.$$

14.2.3 Logarithm of Negative Numbers

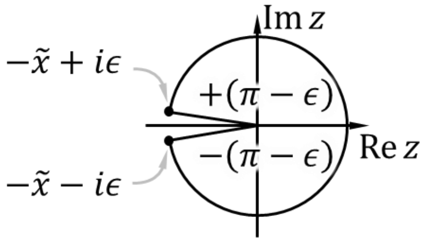
For real numbers x, e^x is always positive and thus $\ln\tilde{x}$ is not defined for $\tilde{x} < 0$. For complex numbers on the other hand, e^z may well be negative, if $\operatorname{Im} z = \pm\pi$, since, for $x \in \mathbb{R}$, $e^{x \pm i\pi} = -e^x < 0$. Thus, we should expect something like $\ln(-e^x) = x \pm i\pi$. Note, that the factor $\pm i\pi$ must be here for any x, thus for any $\tilde{x} = e^x$: We can just as well write

$$\ln(-\tilde{x}) = \ln\tilde{x} + \ln(-1) = \ln\tilde{x} \pm i\pi, \qquad \text{for} \quad \tilde{x} > 0.$$

Unfortunately, we have this disambiguity, that $\ln(-\tilde{x})$ could either be $\ln\tilde{x} + i\pi$ or $\ln\tilde{x} - i\pi$. Thus, the logarithm of a negative real number is still somehow ill defined. Complex numbers with non-vanishing imaginary parts on the order hand do not have this problem, even if the real part is negative. To see this, we add an infinitesimal imaginary part to our negative real number $-\tilde{x}$:

$$-\tilde{x} \pm i\epsilon = \tilde{x}e^{\pm i(\pi-\epsilon)}.$$

This can easily be understood graphically: To reach the point $-\tilde{x} + i\epsilon$ in the complex plane, one can wander about an angle of $+(\pi - \epsilon)$ along a circle with radius $\tilde{x}$. To reach $-\tilde{x} - i\pi$ one wanders in the opposite direction, about the negative angle of $-(\pi - \epsilon)$.



We can convince ourselves also mathematically that this equation holds:

$$\tilde{x}e^{\pm i(\pi-\epsilon)} = \tilde{x}(\cos(\pi+\epsilon) \pm i\sin(\pi+\epsilon)) = \tilde{x}(-\cos\epsilon \mp i\sin\epsilon) = \tilde{x}(-1 \mp i\epsilon) = -\tilde{x} \mp i\tilde{x}\epsilon,$$

where we can replace the infinitesimal quantity $\tilde{x}\epsilon$ again by ϵ. Thus,

$$\ln(-\tilde{x} \pm i\epsilon) = \ln \tilde{x}e^{\pm i(\pi-\epsilon)} = \ln \tilde{x} \pm i(\pi - \epsilon) = \ln \tilde{x} \pm i\pi \qquad \text{or} \qquad \operatorname{Im}\ln(-\tilde{x} \pm i\epsilon) = \pm\pi.$$

The result is the same as the one we wrote above for $\ln(-\tilde{x})$, but now it is well defined, as the choice whether to take plus or minus on the right-hand side is not arbitrary but determined through the left-hand side.

14.2.4 Calculating the Imaginary Part of the Electron Loop

For any fixed $q^2 > 4m^2$, the inequality condition for $\widehat{\Pi}(q^2)$ having an imaginary part,

$$m^2 < x(1-x)q^2,$$

holds for

$$\begin{aligned} &x(1-x) > m^2/q^2 \\ \Leftrightarrow \qquad &x^2 - x + m^2/q^2 < 0. \end{aligned}$$

Replacing $<$ by $=$ we can evaluate the borders of the x-region, where this inequality is fulfilled:

$$x^2 - x + \frac{m^2}{q^2} = 0 \qquad \Longrightarrow \qquad x_\pm = \frac{1}{2} \pm \frac{1}{2}\beta, \qquad \beta \coloneqq \sqrt{1 - 4m^2/q^2}.$$

Now the inequality is fulfilled either outside of those borders or in between, that is either for $x < x_-$ and $x > x_+$ or for $x_- < x < x_+$. The centre of the region $x_- < x < x_+$ is $x = 1/2$, so let's plug this value into the inequality:

$$\underbrace{\frac{1}{2}\left(1 - \frac{1}{2}\right)}_{=1/4} > \frac{m^2}{q^2}.$$

Since we only considered values $q^2 > 4m^2$, the right-hand side is indeed smaller that $1/4$ and the inequality holds. Thus, for $q^2 > 4m^2$, only the region between x_- and x_+ of the integral within $\Pi_R^{(2)}(q^2)$ makes up for an imaginary part.

Using $\operatorname{Im}\ln(-\tilde{x} \pm i\epsilon) = \pm\pi$ (our $-\tilde{x}$ is the large fraction, which therefore luckily completely disappears when calculating the imaginary part) and the substitution $z \coloneqq x - 1/2$, we can evaluate[1]

$$\operatorname{Im}\left(\Pi_R^{(2)}(q^2 \pm i\epsilon)\right) = -\frac{2\alpha_0}{\pi} \operatorname{Im} \int_0^1 dx\, x(1-x) \ln \frac{m^2}{m^2 - x(1-x)(q^2 \pm i\epsilon)}$$

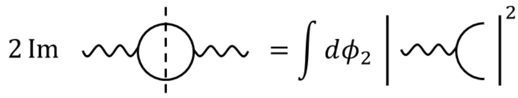
$$= -\frac{2\alpha_0}{\pi} \int_{x_-}^{x_+} dx\, x(1-x) \operatorname{Im} \ln \frac{m^2}{m^2 - x(1-x)q^2 \mp i\epsilon}$$

$$= -\frac{2\alpha_0}{\pi} \int_{x_-}^{x_+} dx\, x(1-x) \operatorname{Im} \ln \left(\frac{m^2}{m^2 - x(1-x)q^2} \pm i\epsilon\right) = -\frac{2\alpha_0}{\pi} (\pm\pi) \int_{\frac{1}{2}-\frac{1}{2}\beta}^{\frac{1}{2}+\frac{1}{2}\beta} dx\, x(1-x)$$

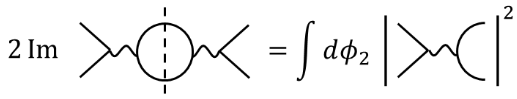
$$= \mp 2\alpha_0 \int_{-\beta/2}^{\beta/2} dz \left(\frac{1}{2} + z\right)\left(\frac{1}{2} - z\right) = \mp 2\alpha_0 \left[\frac{1}{4}z - \frac{1}{3}z^3\right]_{-\beta/2}^{\beta/2} = \mp \frac{\alpha_0}{2} \beta \left(1 - \frac{1}{3}\beta^2\right)$$

$$= \mp \frac{\alpha_0}{3} \sqrt{1 - \frac{4m^2}{q^2}} \left(1 + \frac{2m^2}{q^2}\right).$$

Due to the Cutkosky cutting rules, this imaginary part gives the total amplitude for pair production



Indeed (and without proof), it even gives the correct energy dependence of the total cross section for the more relevant diagram



Let's see what this energy dependence explicitly looks like. Note, that the fermion loop can – in principle – be made up of any fermion (that is, electron, muon or tau). Let's say, we are interested into the electron-muon scattering $e^+e^- \to \mu^+\mu^-$. Then we just need to take the loop fermion to be a muon and replace m by the muon mass m_μ. According to the optical theorem formula for forward scattering from section 12.1, the right-hand side of this picture reads in the centre of mass frame $4|\vec{p}|E_{\text{cm}}\sigma_{\text{tot}}$. Thus, the picture can be written es

$$2 \operatorname{Im} \Pi_R^{(2)}(q^2 \pm i\epsilon) \sim 4|\vec{p}|E_{\text{cm}}\sigma_{\text{tot}},$$

where we have a proportionality instead of an equality, since $\Pi_R^{(2)}$ does not contain the incoming electrons (that they just give a proportionality factor is true, but not proven here). Thus, we have

$$\sigma_{\text{tot}} \sim \frac{1}{2|\vec{p}|E_{\text{cm}}} \operatorname{Im} \Pi_R^{(2)}(q^2 \pm i\epsilon) = \mp \frac{\alpha_0}{6|\vec{p}|E_{\text{cm}}} \sqrt{1 - \frac{4m_\mu^2}{q^2}} \left(1 + \frac{2m_\mu^2}{q^2}\right).$$

Using $|\vec{p}| = \sqrt{E^2 + m^2} \approx E = E_{\text{cm}}/2$ (we neglect the electron mass $m \ll m_\mu$) as well as $q^2 = E_{\text{cm}}^2$, we find

---

[1] We also used

$$\frac{a}{b \mp i\epsilon} = \frac{a}{b(1 \mp i\tilde{\epsilon})} = \frac{a}{b}(1 \pm i\tilde{\epsilon}) = \frac{a}{b} \pm i\tilde{\tilde{\epsilon}}$$

and finally set $\tilde{\tilde{\epsilon}} \to \epsilon$ as usual.

$$\sigma_{\mathrm{tot}}(e^+e^- \to \mu^+\mu^-) \sim \frac{1}{E_{\mathrm{cm}}^2}\sqrt{1-\frac{4m_\mu^2}{E_{\mathrm{cm}}^2}}\left(1+\frac{2m_\mu^2}{E_{\mathrm{cm}}^2}\right).$$

14.3 Momentum-Dependent Effective Charge

14.3.1 Relativistic Limit

Taking the formula for $\Pi_R^{(2)}(q^2)$ (the order-α contribution of $\Pi_R(q^2)$) from section 13.5 and expanding the argument of the logarithm for $-m^2/q^2 \ll 1$ yields, using $x/(x+a) = x/a + \mathcal{O}(x^2)$,

$$\begin{aligned}
\Pi_R^{(2)} &= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\frac{m^2}{-x(1-x)q^2+m^2} \\
&= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\frac{-m^2/q^2}{x(1-x)-m^2/q^2} \\
&\approx -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2}{-q^2}\frac{1}{x(1-x)}\right) \\
&= \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left(\ln\frac{-q^2}{m^2}+\ln x(1-x)\right) \\
&= \frac{2\alpha}{\pi}\left(\ln\frac{-q^2}{m^2}\underbrace{\int_0^1 dx\, x(1-x)}_{=1/6}+\underbrace{\int_0^1 dx\, x(1-x)\,\ln x(1-x)}_{=-5/18}\right) \\
&= \frac{\alpha}{3\pi}\left(\ln\frac{-q^2}{m^2}-\frac{5}{3}\right) \\
&= \frac{\alpha}{3\pi}\ln\frac{-q^2}{m^2 e^{5/3}}.
\end{aligned}$$

Thus, the q^2-dependence coupling constant reads

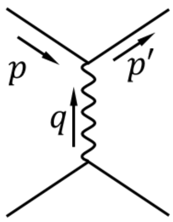
$$\bar{\alpha}(q^2) = \frac{\alpha}{1-\Pi_R(q^2)} \approx \frac{\alpha}{1-\frac{\alpha}{3\pi}\ln\frac{-q^2}{m^2e^{5/3}}}.$$

Growing momentum $-q^2$ means smaller distances means larger coupling constant and hence larger charge.

14.4 Corrections to the Coulomb Potential

14.4.1 Non-Relativistic Limit yields Coulomb Potential

Consider electron-positron scattering, where $q = p' - p$ is the difference between the outgoing and incoming electron momenta.



In the non-relativistic limit, we can write $E = \sqrt{\vec{p}^2+m^2} \approx m$, such that $E' \approx E$ and thus

$$q^2 = (p'-p)^2 = (E'-E)^2 - |\vec{p}'-\vec{p}|^2 \approx -|\vec{p}'-\vec{p}|^2 = -|\vec{q}|^2.$$

Now take a look at the interaction part of this diagram; by “interaction part”, we mean the photon propagator as well as the two vertices ($g = e$):

$$ie\gamma^\mu \frac{-i\eta_{\mu\nu}}{q^2}\, ie\gamma^\nu \approx i\gamma^\mu\gamma_\mu \frac{e^2}{-|\vec{q}|^2} =: i\gamma^\mu\gamma_\mu V(\vec{q}).$$

Let’s compute the Fourier transformation of $V(\vec{q})$ (we use the abbreviation $q \coloneqq |\vec{q}|$ here):

$$\begin{aligned}
V_0(\vec{r}) &= \int d^3\bar{q}\, V(\vec{q})\, e^{i\vec{q}\cdot\vec{r}} = \int d^3\bar{q} \frac{-e^2}{q^2} e^{i\vec{q}\cdot\vec{r}} = -\frac{e^2}{(2\pi)^3}(2\pi)\int_0^\infty q^2\, dq \int_{-1}^1 d\cos\theta \frac{1}{q^2} e^{iqr\cos\theta} \\
&= -\frac{e^2}{(2\pi)^2}\underbrace{\int_0^\infty dq \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)}_{=\pi/q} = -\frac{e^2}{4\pi r} = \frac{-e^2}{4\pi|\vec{r}|}
\end{aligned}$$

That’s just the Coulomb potential!

14.4.2 Contribution to the Lamb Shift

Using the effective charge $e_{\text{eff}}(q^2)$ instead of e, the corrected Coulomb potential reads

$$\begin{aligned}
V(\vec{r}) &= \int d^3\bar{q} \frac{e_{\text{eff}}^2(q^2)}{-|\vec{q}|^2} e^{i\vec{q}\cdot\vec{r}} = \int d^3\bar{q} \frac{e^2}{-|\vec{q}|^2}\frac{1}{1-\Pi_R(q^2)} e^{i\vec{q}\cdot\vec{r}} \\
&= \int d^3\bar{q} \frac{4\pi\alpha}{-|\vec{q}|^2}\left(1 + \Pi_R^{(2)}(q^2) + \mathcal{O}(\alpha^2)\right) e^{i\vec{q}\cdot\vec{r}} \\
&= V_0(\vec{r}) + \int d^3\bar{q} \frac{4\pi\alpha}{-|\vec{q}|^2}\left(\Pi_R^{(2)}(q^2)\right) e^{i\vec{q}\cdot\vec{r}} + \mathcal{O}(\alpha^3).
\end{aligned}$$

Expanding $\Pi_R^{(2)}(q^2) \approx \Pi_R^{(2)}(-|\vec{q}|^2)$ from section 13.5 in the limit $|\vec{q}| \ll m$, using $\ln(1+ax)^{-1} = -\ln(1+ax) \approx -ax$:

$$\begin{aligned}
\Pi_R^{(2)}(-|\vec{q}|^2) &= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x) \ln \frac{1}{1 + x(1-x)\,|\vec{q}|^2/m^2} \\
&\approx \frac{2\alpha}{\pi}\underbrace{\int_0^1 dx\, x^2(1-x)^2}_{=1/30} |\vec{q}|^2/m^2 = \frac{\alpha}{15\pi}\frac{|\vec{q}|^2}{m^2}.
\end{aligned}$$

Thus, the Coulomb potential becomes

$$\begin{aligned}
V(\vec{r}) &= V_0(\vec{r}) + \int d^3\bar{q} \frac{4\pi\alpha}{-|\vec{q}|^2}\left(\frac{\alpha}{15\pi}\frac{|\vec{q}|^2}{m^2}\right) e^{i\vec{q}\cdot\vec{r}} + \mathcal{O}(\alpha^3) \\
&= V_0(\vec{r}) - \frac{4\alpha^2}{15m^2}\int d^3\bar{q}\, e^{i\vec{q}\cdot\vec{r}} + \mathcal{O}(\alpha^3) \\
&= V_0(\vec{r}) - \frac{4\alpha^2}{15m^2}\delta(\vec{r}) + \mathcal{O}(\alpha^3).
\end{aligned}$$

14.4.3 Rewriting the Potential in a More General Case

If we do not want to only consider the limit $|\vec{q}| \ll m$ as in(>14.4.2), it is useful to rewrite the Fourier expansion of the Potential in the following way (we use in this case $q \coloneqq |\vec{q}|$):

$$\begin{aligned}
V(\vec{r}) &= \int d^3\bar{q}\, V(\vec{q})\, e^{i\vec{q}\cdot\vec{r}} \\
&= \frac{1}{(2\pi)^2}\int_0^\infty q^2\, dq\, V(q) \int_{-1}^1 d\cos\theta\ e^{iqr\cos\theta} \\
&= \frac{1}{(2\pi)^2}\int_0^\infty q^2\, dq\, V(q) \frac{e^{iqr} - e^{-iqr}}{iqr}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-i}{(2\pi)^2 r}\left(\int_0^\infty q\,dq\,V(q)e^{iqr} - \int_0^\infty q\,dq\,V(q)\,e^{-iqr}\right) \\
&= \frac{-i}{(2\pi)^2 r}\left(\int_0^\infty q\,dq\,V(q)e^{iqr} + \int_{-\infty}^0 q\,dq\,V(q)\,e^{iqr}\right) \\
&= \frac{-i}{(2\pi)^2 r}\int_{-\infty}^\infty q\,dq\,V(q)e^{iqr} \\
&= \frac{i\alpha}{\pi r}\int_{-\infty}^\infty dq\,\frac{q\,e^{iqr}}{q^2+\nu^2}\frac{1}{1-\Pi_R(-q^2)}.
\end{aligned}
$$

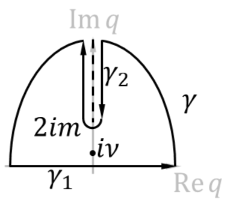
Since $q^2 = 0$ now lies within our integration region $[-\infty, \infty]$, we use a photon mass ν^2 to regulate the integral.

14.4.4 Solving the Integral using Residues Theorem

We want to compute this integral using the residue's theorem. In the complex plane, the integrand is one pole at $q = \pm i\nu$. But also $\Pi_R(-|\vec{q}|^2)$ is not defined for $q^2 > 4m^2$ (with q being again a four-momentum), as we saw in section 14.2. In our non-relativistic limit means, $q^2 > 4m^2$ means (with q being a four-momentum)

$$
q^2 \approx -|\vec{q}|^2 > 4m^2 \qquad \Leftrightarrow \qquad \mathrm{Im}|\vec{q}| > \pm 2m
$$

(*except* for those two equations, we will use $q \coloneqq |\vec{q}|$ henceforth). Thus, the upper-half complex plane of $q = |\vec{q}|$ features one pole at $i\nu$ and one branch cut on the imaginary axis, starting at $q = 2im$.



In this sketch, we already draw a closed contour. We call the whole closed contour γ. The part along the reals axis is called γ_1. We have also two infinitely large quarter-circles, which we do not label by some special γ_i, since they vanish anyway. And then, the contour needs to take a little detour γ_2, since the integrand is not defined at $i\infty$: It needs to avoid the branch cut by going down on the right-hand side of the branch cut and then up again on the left-hand side (γ_2 describes both paths along the branch cut: down and up). Hence, since we assume that the quarter circles vanish, we can write

$$
\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} = 2\pi i \operatorname{res} i\nu \qquad \Leftrightarrow \qquad \int_{\gamma_1} = -\int_{\gamma_2} + 2\pi i \operatorname{res} i\nu.
$$

The integral along γ_1, i. e. the real axis, is just (the integral in) $V(\vec{r})$. Thus, one contribution to $V(\vec{r})$ comes from the residue and one from the detour at the branch cut.

14.4.5 Calculating the Residue (yielding the Coulomb Potential)

First, let's compute the residue. Note, that the denominator of the integrand

$$
f(q) = \frac{i\alpha}{\pi r}\frac{q\,e^{iqr}}{q^2+\nu^2}\frac{1}{1-\Pi_R(-q^2)}
$$

can be written as $q^2 + \nu^2 = (q + i\nu)(q - i\nu)$. Thus, it has two poles, but only one of them lies within γ and we find, using the footnote on page 26,

$$
\begin{aligned}
2\pi i \operatorname{res} i\nu &= 2\pi i (q - i\nu)\, f(q)\big|_{q=i\nu} \\
&= 2\pi i \frac{i\alpha}{\pi r}\frac{q\,e^{iqr}}{(q+i\nu)(q-i\nu)}(q-i\nu)\frac{1}{1-\Pi_R(-q^2)}\bigg|_{q=i\nu}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{2\alpha}{r}\frac{q\, e^{iqr}}{q+i\nu}\frac{1}{1-\Pi_R(-q^2)}\bigg|_{q=i\nu} \\
&= -\frac{2\alpha}{r}\frac{i\nu\, e^{i(i\nu)r}}{(i\nu+i\nu)}\frac{1}{1-\Pi_R(-(i\nu)^2)} \\
&= -\frac{\alpha}{r}e^{-\nu r}\frac{1}{1-\Pi_R(\nu^2)} \\
&= -\frac{\alpha}{r} = V_0(\vec{r}),
\end{aligned}
$$

where we took the limit $\nu \to 0$ in the last step and used $\Pi_R(0) = 0$ from section 13.5. Thus, the residue just gives the Coulomb potential.

14.4.6 Calculating the Branch Cut Contribution (yielding Correction to the Coulomb Potential)

Now to the branch cut: This branch cut is due to the fact that the logarithm in $\Pi_R^{(2)}$ is negative along the imaginary q-axis for $\operatorname{Im} q > 2m$.[1] Thus, our contour γ_2 moves along $q = i\tilde{q} \pm \epsilon$ for $\tilde{q} > 2m$, along which the logarithm takes on the form

$$
\begin{aligned}
\ln\frac{m^2}{m^2 + x(1-x)q^2} &= \ln\frac{m^2}{m^2 + x(1-x)(i\tilde{q} \pm \epsilon)^2} = \ln\frac{m^2}{m^2 - x(1-x)\tilde{q}^2 \pm i\epsilon} \\
&= \ln\left(\frac{m^2}{m^2 - x(1-x)\tilde{q}^2} \mp i\epsilon\right) = \ln\frac{m^2}{x(1-x)\tilde{q}^2 - m^2} \mp i\pi.
\end{aligned}
$$

In the last step we used the formula that we found in section 14.2.

The real part is continuous at the cut; its value is basically on both sides the same. Thus, from the real part point of view, the path γ_2 goes down and then the same way up again, but in the other direction. Hence, the down path and the up path just cancel each other. The real part of $\Pi_R^{(2)}(\nu^2)$ will not contribute to the γ_2 path integral, hence also the real part of $1 + \Pi_R^{(2)}$ will not contribute.

Not so the imaginary part: It is discontinuous on both sides of the cut. Let's abbreviate everything in the integral but $1 + \Pi_R$ by $f(q)$ and evaluate the path integral along γ_2:

$$
\begin{aligned}
\int_{\gamma_2} &:= \frac{i\alpha}{\pi r}\int_{\gamma_2} dq \frac{q\, e^{iqr}}{q^2+\nu^2}\frac{1}{1-\Pi_R(-q^2)} \\
&= \int_{\gamma_2} dq\, f(q)\, \left(1 + \Pi_R^{(2)}(-q^2) + \mathcal{O}(\alpha^2)\right) \\
&= \int_{\gamma_2\downarrow} dq\, f(q)\, \left(1 + \Pi_R^{(2)}(-q^2)\right) + \int_{\gamma_2\uparrow} dq\, f(q)\, \left(1 + \Pi_R^{(2)}(-q^2)\right) + \mathcal{O}(\alpha^3) \\
&= i\int_{\gamma_2\downarrow} dq\, f(q) \operatorname{Im}\Pi_R^{(2)}(-q^2) + i\int_{\gamma_2\uparrow} dq\, f(q) \operatorname{Im}\Pi_R^{(2)}(-q^2) + \mathcal{O}(\alpha^3),
\end{aligned}
$$

where, in the last step, we used our observation that the real parts of the down and up path cancel. We now parameterize the paths using $q = i\tilde{q} \pm \epsilon, \tilde{q} \in [2m, \infty]$, using $-q^2 = \tilde{q}^2 \mp i\epsilon$ (with some new but still infinitesimal ϵ; note, that we can immediately set $\epsilon \to 0$ in $f(q)$)

$$
\int_{\gamma_2} = i\int_{\infty}^{2m} id\tilde{q}\, f(i\tilde{q}) \operatorname{Im}\Pi_R^{(2)}(\tilde{q}^2 - i\epsilon) + i\int_{2m}^{\infty} id\tilde{q}\, f(i\tilde{q}) \operatorname{Im}\Pi_R^{(2)}(\tilde{q}^2 + i\epsilon) + \mathcal{O}(\alpha^3).
$$

[1] For $q = 2im$, we have (for $q := |\vec{q}|$, the two terms in the denominator are added up, not subtracted!)

$$
\ln\frac{m^2}{m^2 + x(1-x)q^2} \overset{q=2im}{=} \ln\frac{1}{1-4x(1-x)}.
$$

Since $x(1-x)$ is at most 1/4, it is obvious, that the argument of the logarithm becomes negative for $\operatorname{Im} q > 2m$ on the imaginary axis.

We know from section (>14.2.4), that

$$\operatorname{Im}\Pi_R^{(2)}(q^2 \pm i\epsilon) = \mp\frac{\alpha}{3}\sqrt{1-\frac{4m^2}{q^2}}\left(1+\frac{2m^2}{q^2}\right) =: \mp\mathcal{I}(q^2).$$

Thus,

$$\int_{\gamma_2} = i\int_{\infty}^{2m} id\tilde{q}\, f(i\tilde{q})\,\mathcal{I}(\tilde{q}^2) - i\int_{2m}^{\infty} id\tilde{q}\, f(i\tilde{q})\,\mathcal{I}(\tilde{q}^2) + \mathcal{O}(\alpha^3) = 2\int_{2m}^{\infty} d\tilde{q}\, f(i\tilde{q})\,\mathcal{I}(\tilde{q}^2) + \mathcal{O}(\alpha^3).$$

Plugging in

$$f(i\tilde{q}) = \frac{i\alpha}{\pi r}\frac{i\tilde{q}\, e^{i(i\tilde{q})r}}{(i\tilde{q})^2+\nu^2} \overset{\nu\to 0}{=} \frac{\alpha}{\pi r}\frac{e^{-\tilde{q}r}}{\tilde{q}},$$

we end up with

$$V(\vec{r}) = \int_{\gamma_1} = 2\pi i \operatorname{res} im - \int_{\gamma_2} = V_0(\vec{r}) - \frac{2\alpha}{\pi r}\int_{2m}^{\infty} d\tilde{q}\,\frac{e^{-\tilde{q}r}}{\tilde{q}}\,\mathcal{I}(\tilde{q}^2) + \mathcal{O}(\alpha^3).$$

14.4.7 The Uehling Potential

Let us compute the order-α^2 correction to the Coulomb potential, that is $\delta V(\vec{r})$ in

$$V(\vec{r}) = V_0(\vec{r}) + \delta V(\vec{r}) + \mathcal{O}(\alpha^3),$$

where

$$\begin{aligned}\delta V(\vec{r}) &= -\frac{2\alpha}{\pi r}\int_{2m}^{\infty} dq\,\frac{e^{-qr}}{q}\,\mathcal{I}(q^2)\\ &= -\frac{2\alpha^2}{3\pi r}\int_{2m}^{\infty} dq\,\frac{e^{-qr}}{q}\sqrt{1-\frac{4m^2}{q^2}}\left(1+\frac{2m^2}{q^2}\right),\end{aligned}$$

in the limit $r \gg 1/m$. In this limit, the integral is dominated by the region where $q \approx 2m$. Let's first substitute $\tilde{q} = q - 2m$,

$$\delta V(\vec{r}) = -\frac{2\alpha^2}{3\pi r}\int_0^{\infty} d\tilde{q}\,\frac{e^{-(\tilde{q}+2m)r}}{\tilde{q}+2m}\sqrt{1-\frac{4m^2}{(\tilde{q}+2m)^2}}\left(1+\frac{2m^2}{(\tilde{q}+2m)^2}\right),$$

and now we can approximate for small $\tilde{q}$:

$$\frac{1}{\tilde{q}+2m} = \frac{1}{2m} + \mathcal{O}(\tilde{q}), \qquad \sqrt{1-\frac{4m^2}{(\tilde{q}+2m)^2}} = \sqrt{\frac{\tilde{q}}{m}} + \mathcal{O}(\tilde{q}), \qquad 1+\frac{2m^2}{(\tilde{q}+2m)^2} = \frac{3}{2} + \mathcal{O}(\tilde{q}).$$

This yields,

$$\begin{aligned}\delta V(\vec{r}) &\approx -\frac{2\alpha^2}{3\pi r}\int_0^{\infty} d\tilde{q}\,\frac{e^{-(\tilde{q}+2m)r}}{2m}\sqrt{\frac{\tilde{q}}{m}}\frac{3}{2}\\ &= -\frac{\alpha^2 e^{-2mr}}{2\pi m^{3/2} r}\underbrace{\int_0^{\infty} d\tilde{q}\, e^{-\tilde{q}r}\sqrt{\tilde{q}}}_{=\sqrt{\pi}/(2r^{3/2})}\\ &= -\frac{\alpha}{r}\,\frac{\alpha e^{-2mr}}{4\sqrt{\pi}(mr)^{3/2}}.\end{aligned}$$

15 FUNCTIONAL INTEGRALS

15.1 Functional Integrals in Quantum Mechanics

15.1.1 Motivation of the Functional Integral Formula

In quantum mechanics, there is a superposition principle: When a process can take place in more than one way, its total amplitude is the coherent sum of the amplitude for each way. In the example of the double-slit experiment, the amplitude of the electron arriving at the screen at one certain position is the sum of the amplitude for the two possible paths through the two possible slits.

None of all the possible paths should be inherently more important than the other; that is, all of them should be included into the calculation with the same weight. Or in other words: All paths are equally probable. The path through the left slit is no more probable than the path through the right slit. That is, the absolute squared of the amplitudes of the different paths are equal. However, the paths differ in phase and when we add up the two possible paths, we might get interference. Therefore, we write each path only as a phase ϕ:

$$U(x_a, x_b, T) = \sum_{\text{all paths } x(t)} e^{i\phi[x(t)]} =: \int \mathcal{D}x(t)\, e^{i\phi[x(t)]}.$$

Of course, only paths $x(t)$ which start at x_a and end at x_b are considered. Depending on the path the electron takes, it has a phase $\phi[x(t)]$ in the end. Thus, the phase is a value assigned to a path; that is, $\phi[x(t)]$ is a functional. And since paths are in general not discrete but continuous, we write an integral $\int \mathcal{D}x(t)$ instead of a sum over "all paths".

A functional can also be differentiated with respect to is argument (a function) and we will write this functional derivative as $\delta\phi[x(t)]/\delta x(t)$.

The remaining question is, how the phase $\phi[x(t)]$ looks like. In the classical limit, the electron should take only a single path $x_{\text{cl}}(t)$. That is, in that limit, only this path contributes to the amplitude. All other paths need to cancel in the sum/integral above. By the principle of the stationary phase, we conclude that the only phase, which is not cancelled, is the one which does not change is the one that obeys $\delta\phi[x(t)]/\delta x(t) = 0$. Hence, the classical path $x_{\text{cl}}(t)$ obeys

$$\left.\frac{\delta\phi[x(t)]}{\delta x(t)}\right|_{x_{\text{cl}}(t)} = 0.$$

There is also the principle of the least action, that tells us, that the classical path satisfies

$$\left.\frac{\delta S[x(t)]}{\delta x(t)}\right|_{x_{\text{cl}}(t)} = 0,$$

where $S = \int dt\, L$ is the classical action. Therefore, it seems reasonable to take $\phi \sim S$. Obviously, the phase in dimensionless. Thus, we should add a factor of dimension action^{-1}. What else could it be than the quantum of action $\hbar$? Thus, in SI units we write $\phi = S/\hbar$ and in natural units simply $\phi = S$.

15.1.2 Verification of the Functional Integral Formula for the Double-Slit Experiment

Let's verify that we can take the phase proportional to the action for the example of the double-slit experiment. Consider the detector to be at a position such that the path from one slit to the detector has length D and the other path is about d longer, that is it has length $D + d$. Let's assume that we send out a single electron to the double-slit at receive it after a time t in the detector. The action for the first and second path then is

$$S_1 = \frac{m}{2} v_1^2 t, \qquad \text{with} \qquad v_1 = D/t,$$

$$S_2 = \frac{m}{2} v_2^2 t, \qquad \text{with} \qquad v_2 = (D+d)/t.$$

Assuming $d \ll D$ and plugging in the velocities, we find

$$S_1 = \frac{m}{2}\frac{D^2}{t}, \qquad S_2 = \frac{m}{2}\frac{(D+d)^2}{t} \approx \frac{m}{2}\frac{D^2 + 2Dd}{t}.$$

The excess phase of the second path is obviously

$$\Delta\phi = \frac{\Delta S}{\hbar} = \frac{d}{\hbar}\underbrace{\frac{mD}{t}}_{\approx p} = 2\pi\frac{pd}{h} = 2\pi\frac{d}{\lambda}.$$

That is, for $d = n\lambda$ we find $e^{i\Delta\phi} = 1$ and thus constructive interference, as we would expect.

15.1.3 Derivation of the Functional Integral Formula

We now want to proof the equality

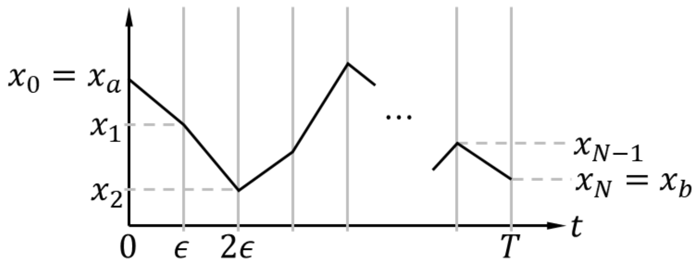
$$U(x_a, x_b, T) := \langle x_b | e^{-iHT} | x_a \rangle = \int \mathcal{D}x(t)\, e^{iS[x(t)]}$$

more rigorously. $U(x_a, x_b, T)$ describes the amplitude of a particle travelling from x_a to x_b in time T. We will show the equality by discretization: We break up the time T into N intervals of duration ϵ and finally perform the limit $N \to \infty, \epsilon \to 0$. In each interval of duration ϵ, we will approximate the path $x(t)$ as a straight line.

The action integral for the discretized path $x(t)$ can be given as a sum over all the time intervals:

$$S = \int_0^T dt\, L = \int_0^T dt\, \left(\frac{m}{2}\dot{x}^2 + V(x)\right) = \sum_{n=0}^{N-1} \epsilon \left(\frac{m}{2}\left(\frac{x_{n+1} - x_n}{\epsilon}\right)^2 - V\left(\frac{x_{n+1} + x_n}{2}\right)\right).$$

The time slices are referenced by an index n. We use $x_0 := x_a$ and $x_N := x_b$. Since ϵ is the length of an infinitesimal time interval, we can use $dt = \epsilon$ as well as $\dot{x} = (x_{n+1} - x_n)/\epsilon$ (that is just the slope of the straight path within the n-th interval). For the argument of the potential V, we just use the average position $(x_{n+1} + x_n)/2$ within the given time interval.



So far, we only considered the action. However, the action needs to be plugged into a functional integral. We define the functional integral to be integrals over the positions $x_1, x_2, \ldots, x_{N-1}$. As these positions are always connected by straight lines, the tuple $(x_1, x_2, \ldots, x_{N-1})$ will – together with the fixed start and end points $x_0 = x_a, x_N = x_b$ – define a path entirely. As it turns out later, we also should include some ϵ-dependent constant a_ϵ^{-1} for each of the N intervals:

$$\int \mathcal{D}x := \frac{1}{a_\epsilon}\int_{-\infty}^{\infty} \frac{dx_1}{a_\epsilon}\frac{dx_2}{a_\epsilon}\cdots\frac{dx_{N-1}}{a_\epsilon}.$$

With this expression, we can write

$$U(x_0,x_N,T) = \underbrace{\frac{1}{a_\epsilon}\int_{-\infty}^{\infty}\frac{dx_1}{a_\epsilon}\frac{dx_2}{a_\epsilon}\cdots\frac{dx_{N-1}}{a_\epsilon}}_{=\int\mathcal{D}x}\underbrace{\prod_{n=0}^{N-1}\exp\left(i\frac{m}{2}\frac{(x_{n+1}-x_n)^2}{\epsilon}-i\epsilon V\left(\frac{x_{n+1}+x_n}{2}\right)\right)}_{=e^{iS[x(t)]}}$$
$$= \int_{-\infty}^{\infty}\frac{dx_{N-1}}{a_\epsilon}\exp\left(i\frac{m}{2}\frac{(x_N-x_{N-1})^2}{\epsilon}-i\epsilon V\left(\frac{x_N+x_{N-1}}{2}\right)\right)U(x_0,x_{N-1},T-\epsilon).$$

In the last step, we put all integrals except for the dx_{N-1}-integral and all factors of the large product except the one of the last time interval into $U(x_0,x_{N-1},T-\epsilon)$.

In the limit $\epsilon\to 0$ the first term of the exponent oscillates very rapidly and x_N must be very close to x_{N-1} to not be cancelled out by this oscillation $x_{N-1}\approx x_N$. Thus, we only get x_N in the argument of the potential and we write x_N instead of x_{N-1} as the second argument of the U beneath the integral:

$$U(x_0,x_N,T) = \int_{-\infty}^{\infty}\frac{dx_{N-1}}{a_\epsilon}\exp\left(i\frac{m}{2\epsilon}(x_N-x_{N-1})^2-i\epsilon V(x_N)\right)U(x_0,x_{N-1},T-\epsilon).$$

And we can also expand the amplitude $U(x_0,x_{N-1},T-\epsilon)=:\widetilde{U}(x_{N-1})$. We take it as a function of x_{N-1} and Taylor expand it at $x_{N-1}\approx x_N$:

$$\widetilde{U}(x_{N-1}) = \widetilde{U}(x_N)+\frac{\partial}{\partial x_{N-1}}\widetilde{U}(x_{N-1})\bigg|_{x_N}(x_{N-1}-x_N)+\frac{1}{2}\frac{\partial^2}{\partial x_{N-1}^2}\widetilde{U}(x_{N-1})\bigg|_{x_N}(x_{N-1}-x_N)^2+\cdots$$
$$=\left(1+(x_{N-1}-x_N)\frac{\partial}{\partial x_N}+\frac{1}{2}(x_{N-1}-x_N)^2\frac{\partial^2}{\partial x_N^2}+\cdots\right)\widetilde{U}(x_N).$$

If we also expand $\exp(-i\epsilon V)\approx 1-i\epsilon V+\mathcal{O}(\epsilon^2)$, we find

$$U(x_0,x_N,T) = \int_{-\infty}^{\infty}\frac{dx_{N-1}}{a_\epsilon}\exp\left(i\frac{m}{2\epsilon}(x_N-x_{N-1})^2\right)(1-i\epsilon V+\cdots)$$
$$\cdot\left(1+(x_{N-1}-x_N)\frac{\partial}{\partial x_N}+\frac{1}{2}(x_{N-1}-x_N)^2\frac{\partial^2}{\partial x_N^2}+\cdots\right)U(x_0,x_N,T-\epsilon).$$

If we shift the integration variable $x_{N-1}\to x_{N-1}+x_N=:\xi$, the integrals dx_{N-1} are now nothing else but Gaussian integrals[1]

$$\int d\xi\ e^{-a\xi^2}=\sqrt{\frac{\pi}{a}},\qquad \int d\xi\ \xi\ e^{-a\xi^2}=0,\qquad \int d\xi\ \xi^2\ e^{-a\xi^2}=\frac{1}{2a}\sqrt{\frac{\pi}{a}}.$$

Thus, we find

[1] Well, not quite. Gaussian integrals usually have *real* parameters a, where as in our case, a is purely imaginary. Actually,

$$\int d\xi\ e^{-a\xi^2}=\sqrt{\frac{\pi}{a}},$$

does hold also for complex a, as long as $\mathrm{Re}\,a>0$ (we won't proof this mathematical statement). Thus, we can write

$$\int d\xi\ e^{-ib\xi^2}=\int d\xi\ e^{-i(b-i\epsilon)\xi^2}=\int d\xi\ e^{-(ib+\epsilon)\xi^2}=\sqrt{\frac{\pi}{ib+\epsilon}}=\sqrt{\frac{-i\pi}{b-i\epsilon}},$$

the limit $\epsilon\to 0$ of which gives the integral for a purely imaginary $a=ib$. If b is never zero, we can obviously simply set $\epsilon=0$. However, it will happen that b is something like p^2-m^2 and then we need the $i\epsilon$, since p^2-m^2 can be zero.

$$
\begin{aligned}
&U(x_0, x_N, T) \\
&\quad = \int_{-\infty}^{\infty} \frac{d\xi}{a_\epsilon} \exp\left(-\frac{m}{2i\epsilon}\xi^2\right)(1 - i\epsilon V + \cdots)\left(1 + \xi\frac{\partial}{\partial x_N} + \frac{1}{2}\xi^2\frac{\partial^2}{\partial x_N^2} + \cdots\right)U(x_0, x_N, T - \epsilon) \\
&\quad = \frac{1}{a_\epsilon}(1 - i\epsilon V + \cdots)\sqrt{\frac{2\pi i\epsilon}{m}}\left(1 + \frac{1}{2}\frac{2i\epsilon}{2m}\frac{\partial^2}{\partial x_N^2} + \cdots\right)U(x_0, x_N, T - \epsilon) \\
&\quad = \frac{1}{a_\epsilon}\sqrt{\frac{2\pi i\epsilon}{m}}\left(1 - i\epsilon V + i\epsilon\frac{1}{2m}\frac{\partial^2}{\partial x_N^2} + \mathcal{O}(\epsilon^2)\right)U(x_0, x_N, T - \epsilon).
\end{aligned}
$$

In the limit $\epsilon \to 0$, only the bracket becomes 1 and the latter U equal to the U on the left-hand side. Thus, for the equation to hold in this limit, we need

$$
a_\epsilon = \sqrt{\frac{2\pi i\epsilon}{m}}.
$$

Plugging in this expression for a_ϵ and rearranging the equation a little bit, we find that we can give it in the form

$$
\begin{aligned}
&i\frac{U(x_0, x_N, T) - U(x_0, x_N, T - \epsilon)}{\epsilon} = \left(-\frac{1}{2m}\frac{\partial^2}{\partial x_N^2} + V\right)U(x_0, x_N, T - \epsilon) \\
&\Rightarrow \qquad i\frac{\partial}{\partial T}U(x_a, x_b, T) = \left(-\frac{1}{2m}\frac{\partial^2}{\partial x_b^2} + V\right)U(x_a, x_b, T) = HU(x_a, x_b, T).
\end{aligned}
$$

What we have now proofed, is that $U(x_a, x_b, T) = \int \mathcal{D}x(t)\, e^{iS[x(t)]}$ obeys the Schrödinger equation. And so does $U(x_a, x_b, T) = \langle x_b | e^{-iHT} | x_a \rangle$, as it is very easy to see:

$$
i\frac{\partial}{\partial T}U(x_a, x_b, T) = i\frac{\partial}{\partial T}\langle x_b | e^{-iHT} | x_a \rangle = H\langle x_b | e^{iHT} | x_a \rangle = HU(x_a, x_b, T).
$$

Does the fact that $\int \mathcal{D}x(t)\, e^{iS[x(t)]}$ and $\langle x_b | e^{-iHT} | x_a \rangle$ obey the Schrödinger equation mean that they are equal (as was our aim to show)? It *almost* does. If they both obey the same differential equation *and the same initial condition*, then they must be equal. Let's check the initial conditions. For $T \to 0$, we find

$$
\langle x_b | e^{-iHT} | x_a \rangle \xrightarrow{T \to 0} \langle x_b | x_a \rangle = \delta(x_b - x_a).
$$

When it comes to $\int \mathcal{D}x(t)\, e^{iS[x(t)]}$, we can approximate it in the limit $T \to 0$ by a single time interval with a straight line from x_a to x_b:

$$
\int \mathcal{D}x(t)\, e^{iS[x(t)]} \xrightarrow{T \to 0} \sqrt{\frac{m}{2\pi i\epsilon}} \exp\left(i\frac{m}{2}\frac{(x_b - x_a)^2}{\epsilon} + \mathcal{O}(\epsilon)\right).
$$

Note, that the prefactor is just $1/a_\epsilon$, one of which we introduced for each time interval. We want this expression to be a δ-function $\delta(x_b - x_a)$, thus we must examine its behaviour underneath an integral together with a test function:

$$
\int dx\, f(x)\sqrt{\frac{m}{2\pi i\epsilon}} \exp\left(i\frac{m}{2}\frac{x^2}{\epsilon}\right).
$$

As $\epsilon \to 0$, the exponential function oscillates infinitely fast under the variation of x. That is, even if we vary x but an amount δx so small, that f is virtually constant in this region, $f(x + \delta x) \approx f(x)$, the exponential function will have oscillated a lot in this region cancelling out any contribution to the integral. Only for $x = 0$, there is no oscillation no matter how small ϵ is. Therefore, only $x = 0$

contributes and the integral is proportional to $\delta(x)$. We will not proof that the proportionality factor is indeed 1.

15.1.4 Functional Integral Formula for a General Hamiltonian

Let us now consider a quantum system described by an arbitrary set of coordinates q^i and conjugate momenta p^i (the standard-case would be $i = x, y, z$), such that $\vec{q} = (q^1, q^2, \dots)$. The transition amplitude now reads

$$U(\vec{q}_a, \vec{q}_b, T) \coloneqq \langle \vec{q}_b | e^{-iHT} | \vec{q}_a \rangle.$$

As in (>15.1.3) we break up the time interval T into N time intervals of duration ϵ, such that $e^{-iHT} = (e^{-iH\epsilon})^N$. If we then plug in a complete set of states

$$\mathbb{I} = \int \left(\Pi_i dq^i\right) |\vec{q}\rangle\langle\vec{q}|$$

between each of those factors, we find

$$\begin{aligned} U(\vec{q}_0, \vec{q}_N, T) &\coloneqq \int \left(\Pi_i \Pi_{n=1}^{N-1} dq_n^i\right) \langle \vec{q}_N | e^{-iH\epsilon} | \vec{q}_{N-1} \rangle \langle \vec{q}_{N-1} | e^{-iH\epsilon} | \vec{q}_{N-2} \rangle \cdots \langle \vec{q}_1 | e^{-iH\epsilon} | \vec{q}_0 \rangle \\ &= \int \left(\Pi_i \Pi_{n=1}^{N-1} dq_n^i\right) \prod_{m=0}^{N-1} \langle \vec{q}_{m+1} | e^{-iH\epsilon} | \vec{q}_m \rangle. \end{aligned}$$

where we used $\vec{q}_0 \coloneqq \vec{q}_a, \vec{q}_N \coloneqq \vec{q}_b$. Let's assume the Hamiltonian is a sum of terms depending on the coordinates and the momenta respectively: $H(\vec{q}, \vec{p}) = f(\vec{q}) + f(\vec{p})$. Pick one of the many matrix elements, expand in ϵ and plug in this form of the Hamiltonian:

$$\langle \vec{q}_{n+1} | e^{-iH\epsilon} | \vec{q}_n \rangle = \langle \vec{q}_{n+1} | 1 - iH(\vec{q}, \vec{p})\epsilon + \cdots | \vec{q}_n \rangle = \langle \vec{q}_{n+1} | 1 - if(\vec{q})\epsilon - if(\vec{p})\epsilon + \cdots | \vec{q}_n \rangle.$$

A matrix element with $f(\vec{q})$ can be written as

$$\langle \vec{q}_{n+1} | f(\vec{q}) | \vec{q}_n \rangle = f(\vec{q}_n)\, \delta(\vec{q}_{n+1} - \vec{q}_n) = f(\vec{q}_{n+1,n}) \int \left(\Pi_i \frac{dp^i}{2\pi}\right) \exp\left(i\vec{p}_n(\vec{q}_{n+1} - \vec{q}_n)\right).$$

In the second step we used the presence of the δ-function to turn $\vec{q}_n$ into $\vec{q}_{n+1,n} \coloneqq (\vec{q}_{n+1} + \vec{q}_n)/2$ in the argument of f and then wrote the δ-function as an integral, using $\delta(x) = \int dp/2\pi\ e^{ipx}$ (see also the footnote on page 21).

When it comes to the matrix element with $f(\vec{p})$, we can introduce a complete set of momentum states and write

$$\begin{aligned} \langle \vec{q}_{n+1} | f(\vec{p}) | \vec{q}_n \rangle &= \int \left(\Pi_i dp_n^i\right) \langle \vec{q}_{n+1} | f(\vec{p}) | \vec{p}_n \rangle \langle \vec{p}_n | \vec{q}_n \rangle = \int \left(\Pi_i dp_n^i\right) f(\vec{p}_n)\, \langle \vec{q}_{n+1} | \vec{p}_n \rangle \langle \vec{p}_n | \vec{q}_n \rangle \\ &= \int \left(\Pi_i \frac{dp_n^i}{2\pi}\right) f(\vec{p}_n) \exp\left(i\vec{p}_n(\vec{q}_{n+1} - \vec{q}_n)\right). \end{aligned}$$

Finally, also the term $\langle \vec{q}_{n+1} | 1 | \vec{q}_n \rangle$ from the expansion of the Hamiltonian can be written similarly (just plug in $f(\vec{p}) = 1$) in the derivation above) and we find

$$\begin{aligned} \langle \vec{q}_{n+1} | e^{-iH\epsilon} | \vec{q}_n \rangle &= \int \left(\Pi_i \frac{dp_n^i}{2\pi}\right) \left(1 - if(\vec{q}_{n+1,n})\epsilon - if(\vec{p}_n)\epsilon + \cdots\right) \exp\left(i\vec{p}_n(\vec{q}_{n+1} - \vec{q}_n)\right) \\ &= \int \left(\Pi_i \frac{dp_n^i}{2\pi}\right) \exp\left(-i\epsilon H(\vec{q}_{n+1,n}, \vec{p}_n)\right) \exp\left(i\vec{p}_n(\vec{q}_{n+1} - \vec{q}_n)\right). \end{aligned}$$

Plugging this result into U gives

$$
\begin{aligned}
U(\vec{q}_0,\vec{q}_N,T) &= \int \left(\Pi_i \Pi_{n=1}^{N-1} dq_n^i\right) \prod_{m=0}^{N-1} \int \left(\Pi_i \frac{dp_m^i}{2\pi}\right) \exp\left(-i\epsilon H\left(\vec{q}_{m+1,m},\vec{p}_m\right)\right) \exp\left(i\vec{p}_m(\vec{q}_{m+1}-\vec{q}_m)\right) \\
&= \int \left(\Pi_i \left(\Pi_{n=1}^{N-1} dq_n^i\right) \left(\Pi_{m=0}^{N-1} \frac{dp_m^i}{2\pi}\right)\right) \exp\left(i \sum_{m=0}^{N-1} \epsilon \left(\vec{p}_m \frac{\vec{q}_{m+1}-\vec{q}_m}{\epsilon} - H\left(\vec{q}_{m+1,m},\vec{p}_m\right)\right)\right) \\
&= \int \mathcal{D}\vec{q}(t)\, \mathcal{D}\vec{p}(t) \exp\left(i \int_0^T dt \left(\vec{p}\dot{\vec{q}} - H(\vec{q},\vec{p})\right)\right).
\end{aligned}
$$

In the last step we took the limit $\epsilon \to 0$ and changed the discrete to a continuous form. Note, that the coordinate integral $\mathcal{D}\vec{q}$ only includes $N-1$ integrals dq_n^i from $n=1$ to $n=N-1$, as the initial and the final coordinates a fixed by $\vec{q}_0 = \vec{q}_a$ and $\vec{q}_N = \vec{q}_b$. However, the momentum integral $\mathcal{D}\vec{p}$ contains N integrals dp_n^i from $n=0$ to $n=N-1$, since the initial and final momenta are not fixed. Note, that the integrand does not contain a momentum $\vec{p}_N$, which is why there is also no such integration. Obviously, we have defined

$$
\mathcal{D}\vec{q} = \Pi_i \Pi_{n=1}^{N-1} dq_n^i, \qquad \mathcal{D}\vec{p} = \Pi_i \Pi_{m=0}^{N-1} \frac{dp_m^i}{2\pi}.
$$

15.1.5 Non-Relativistic Limit of the General Formula

Let us now examine, if we can extract the non-relativistic limit from this general functional integral formula. In this (one dimensional) case, the Hamiltonian is simply $H = p^2/2m + V(q)$. If we plug this into the discretized functional integral formula (see the end of (>15.1.4)), it reads

$$
\begin{aligned}
&U(\vec{q}_0,\vec{q}_N,T) \\
&\quad = \int \left(\left(\Pi_{n=1}^{N-1} dq_n\right) \left(\Pi_{m=0}^{N-1} \frac{dp_m}{2\pi}\right)\right) \exp\left(i \sum_{m=0}^{N-1} \epsilon \left(p_m \frac{q_{m+1}-q_m}{\epsilon} - \frac{p_m^2}{2m} - V\left(q_{m+1,m}\right)\right)\right).
\end{aligned}
$$

Recall, that $q_{m+1,m} \coloneqq (q_{m+1}+q_m)/2$. We can now explicitly evaluate one of the momentum integrals by completing the square:

$$
\begin{aligned}
&\int \frac{dp_m}{2\pi} \exp\left(ip_m(q_{m+1}-q_m) - i\epsilon \frac{p_m^2}{2m}\right) \\
&\quad = \int \frac{dp_m}{2\pi} \exp\left(-i\left(\sqrt{\epsilon/2m}\, p_m - \frac{1}{2}\sqrt{2m/\epsilon}\,(q_{m+1}-q_m)\right)^2 + i\frac{1}{4}\frac{2m}{\epsilon}(q_{m+1}-q_m)^2\right) \\
&\quad = \exp\left(i\frac{m}{2\epsilon}(q_{m+1}-q_m)^2\right) \int \frac{dp_m}{2\pi} \exp\left(-i\epsilon \frac{p_m^2}{2m}\right) = \exp\left(i\frac{m}{2\epsilon}(q_{m+1}-q_m)^2\right) \frac{1}{2\pi} \sqrt{\frac{2\pi m}{i\epsilon}} \\
&\quad = \frac{1}{a_\epsilon} \exp\left(i\epsilon \frac{m}{2}\left(\frac{q_{m+1}-q_m}{\epsilon}\right)^2\right),
\end{aligned}
$$

where we used the Gaussian integral and $a_\epsilon = \sqrt{2\pi i\epsilon/m}$, both given in (>15.1.3). Since we have N integrals dp_m (from $m=0$ to $m=N-1$), we can N factors a_ϵ^{-1}. If we plug this back into $U(\vec{q}_0,\vec{q}_N,T)$, we arrive at precisely the same expression that we had quite in the beginning of (>15.1.3):

$$
\begin{aligned}
U(\vec{q}_0,\vec{q}_N,T) &= \frac{1}{a_\epsilon^N} \int \left(\Pi_{n=1}^{N-1} dq_n\right) \exp\left(i \sum_{m=0}^{N-1} \epsilon \left(\frac{m}{2}\left(\frac{q_{m+1}-q_m}{\epsilon}\right)^2 - V\left(q_{m+1,m}\right)\right)\right) \\
&= \underbrace{\frac{1}{a_\epsilon} \int \left(\Pi_{n=1}^{N-1} \frac{dq_n}{a_\epsilon}\right)}_{=\int \mathcal{D}x} \underbrace{\prod_{m=0}^{N-1} \exp\left(\left(i\frac{m}{2}\frac{(q_{m+1}-q_m)^2}{\epsilon} - i\epsilon V\left(q_{m+1,m}\right)\right)\right)}_{=e^{iS[x(t)]}}.
\end{aligned}
$$

15.2 Quantization of Scalar Fields

15.2.1 Matrix Element in Terms of the Lagrangian Density

In section 3.1 we introduced the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi) - V(\phi), \qquad V(\phi) = \frac{m^2}{2}\phi^2,$$

as the one whose Euler-Lagrange equation is the Klein-Gordon equation. We then found in section 3.3 that from this Lagrangian density the following Hamiltonian follows:

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right),$$

where we used the conjugate momentum $\Pi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$. $\mathcal{H}$ is by definition the Hamiltonian density. Thus, our functional integral formula from section 15.1 becomes in the case of field theory

$$\langle \phi_b(\vec{x})|e^{-iHT}|\phi_b(\vec{x})\rangle = \int \mathcal{D}\phi\, \mathcal{D}\Pi \exp\left(i\int_0^T d^4x\,(\Pi\dot{\phi} - \mathcal{H})\right)$$
$$= \int \mathcal{D}\phi\, \mathcal{D}\Pi \exp\left(i\int_0^T d^4x \left(\Pi\dot{\phi} - \frac{1}{2}\Pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right)\right),$$

where the $\mathcal{D}\phi$ integral covers all possible fields, which obey $\phi(\vec{x},0) = \phi_a(\vec{x})$ and $\phi(\vec{x},T) = \phi_b(\vec{x})$. If we complete the square with respect to Π, we can evaluate the $\mathcal{D}\Pi$ integral, such that

$$\langle \phi_b(\vec{x})|e^{-iHT}|\phi_b(\vec{x})\rangle = \int \mathcal{D}\phi\, \mathcal{D}\Pi \exp\left(i\int_0^T d^4x \left(-\frac{1}{2}\left(\Pi - \dot{\phi}\right)^2 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right)\right)$$
$$\sim \int \mathcal{D}\phi \exp\left(i\int_0^T d^4x \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right)\right).$$

The result of the $\mathcal{D}\Pi$ integration is some number, which we did not write explicitly here. We will from now on absorb it into the measure $\mathcal{D}\phi$ (after the absorption, we can write an equal sign instead of the proportionality sign). Using $\dot{\phi}^2 - (\nabla\phi)^2 = (\partial^\mu\phi)^2$ we find

$$\langle \phi_b(\vec{x})|e^{-iHT}|\phi_b(\vec{x})\rangle = \int \mathcal{D}\phi \exp\left(i\int_0^T d^4x \left(\frac{1}{2}(\partial^\mu\phi)^2 - V(\phi)\right)\right) = \int \mathcal{D}\phi \exp\left(i\int_0^T d^4x\, \mathcal{L}\right).$$

15.2.2 2-Point Function in Terms of Functional Integrals

We will proof this formula for n-point now for the case $n = 2$:

$$\langle\Omega|\mathcal{T}\phi_H(x_1)\phi_H(x_2)|\Omega\rangle = \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\exp(i\int d^4x\, \mathcal{L})}{\int \mathcal{D}\phi\, \exp(i\int d^4x\, \mathcal{L})}.$$

Let's start with the numerator of the right-hand side. It contains a functional integral covering all possible fields $\phi(x)$ (they should all be fixed at $t = -\infty$ and $t = \infty$). We can break up the path integral as follows:

$$\int \mathcal{D}\phi\, f(\phi) = \int \mathcal{D}\phi_1\, \mathcal{D}\phi_2 \int \mathcal{D}\phi\, \delta[\phi(\vec{x}, x_1^0) - \phi_1(\vec{x})]\, \delta[\phi(\vec{x}, x_2^0) - \phi_2(\vec{x})]\, f(\phi).$$

That is to say, we constrain the fields of the $\mathcal{D}\phi$ integral in such a way, that they must equal $\phi_1(\vec{x})$ at the time x_1^0 and they must equal $\phi_2(\vec{x})$ at the time x_2^0. Then we integrate over all the possible constraints $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$, such that the right-hand side just equals the general integral $\mathcal{D}\phi$ over all possible fields ϕ without any constraints.

Since those "δ-functions" ensure that it is true, we can write $\phi(x_1) \equiv \phi(\vec{x}_1, x_1^0) = \phi_1(\vec{x}_1)$ and similarly $\phi(x_2) = \phi_2(\vec{x}_2)$ and therefore take them out of the $\mathcal{D}\phi$ integral:

$$\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\exp\left(i\int d^4x\,\mathcal{L}\right)$$
$$= \int \mathcal{D}\phi_1\,\mathcal{D}\phi_2\ \phi_1(\vec{x}_1)\ \phi_2(\vec{x}_2)$$
$$\int \mathcal{D}\phi\ \delta[\phi(\vec{x},x_1^0)-\phi_1(\vec{x})]\ \delta[\phi(\vec{x},x_2^0)-\phi_2(\vec{x})]\ \exp\left(i\int d^4x\,\mathcal{L}\right).$$

Recall that the integration $\mathcal{D}\phi$ includes fields with constraints ϕ_a and ϕ_b for $t=\mp\infty$. The "δ-functions" impose *two additional* constraints: ϕ_1 at $t=x_1^0$ and ϕ_2 at $t=x_2^0$.

To understand how we can use those constraints, let's consider an analogous one-dimensional case. Consider the functional integral

$$\int \mathcal{D}f\ e^{\int_{-\infty}^{\infty}dx\,A[f(x)]}\ \delta[f(x_1)-f_1]\ \delta[f(x_2)-f_2]$$
$$= \int \mathcal{D}f\ e^{\int_{-\infty}^{x_1}dx\,A[f(x)]}e^{\int_{x_1}^{x_2}dx\,A[f(x)]}e^{\int_{x_2}^{\infty}dx\,A[f(x)]}\ \delta[f(x_1)-f_1]\ \delta[f(x_2)-f_2],$$

where $0<x_1<x_2<1$. We also assume that the boundaries of the functional integral, $f(-\infty)$ and $f(\infty)$, are fixed. In the step we already made here, we simply split the integral in the exponential. We obviously can always do this (so far, we could have also cut the integral at other places than x_1 and x_2). Our x here plays the role of the time in our field theory calculation. We just don't bother with other (space) coordinate here. Since the regions do not overlap, and since we integrate over all possible functions, we could just as well write

$$\int \mathcal{D}f\,\mathcal{D}h\,\mathcal{D}g\ e^{\int_{-\infty}^{x_1}dx\,A[f(x)]}e^{\int_{x_1}^{x_2}dx\,A[h(x)]}e^{\int_{x_2}^{\infty}dx\,A[g(x)]}\ \delta[\cdots]\cdots.$$

We just need a bunch of more "δ-functions", to ensure that, for example, $f(x_1)=f_1$ or $g(x_2)=f_2$ (they are indicated by the dots). Obviously, those integrals factor out. Since now the "δ-function" only fix the boundaries of integration (namely x_1 is a boundary of $\int_{-\infty}^{x_1}dx\,A[f(x)]$ or x_2 is a boundary of $\int_{x_2}^{\infty}dx\,A[g(x)]$, for example) and since functional integrals usually come with fixed boundaries write

$$\int \mathcal{D}f\ e^{\int_{-\infty}^{x_1}dx\,A[f(x)]}\int \mathcal{D}h\ e^{\int_{x_1}^{x_2}dx\,A[h(x)]}\int \mathcal{D}g\ e^{\int_{x_2}^{\infty}dx\,A[g(x)]}$$

without "δ-functions" and take the fixed values as the fixed boundaries (which we never explicitly denoted in formulas of functional integrals).

In the same way, we get three distinct factors of functional integrals:

$$\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\exp\left(i\int d^4x\,\mathcal{L}\right)$$
$$= \int \mathcal{D}\phi_1\,\mathcal{D}\phi_2\ \phi_1(\vec{x}_1)\ \phi_2(\vec{x}_2)$$
$$\int \mathcal{D}\phi\exp\left(i\int d^4x\,\mathcal{L}[\phi]\right)\int \mathcal{D}\phi'\exp\left(i\int d^4x\,\mathcal{L}[\phi']\right)\int \mathcal{D}\phi''\exp\left(i\int d^4x\,\mathcal{L}[\phi'']\right).$$

The $\mathcal{D}\phi$ integral has boundaries ϕ_a and ϕ_1, the $\mathcal{D}\phi'$ integral has boundaries ϕ_1 and ϕ_2 and the $\mathcal{D}\phi''$ integral has boundaries ϕ_2 and ϕ_b. Using the formula derived in (>15.2.1), we can write them as transition amplitudes

$$\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\exp\left(i\int d^4x\,\mathcal{L}\right)$$
$$= \lim_{T\to\infty}\int \mathcal{D}\phi_1\,\mathcal{D}\phi_2\ \phi_1(\vec{x}_1)\ \phi_2(\vec{x}_2)\,\langle\phi_b|e^{-iH(T-x_2^0)}|\phi_2\rangle\langle\phi_2|e^{-iH(x_2^0-x_1^0)}|\phi_1\rangle\langle\phi_1|e^{-iH(x_1^0-(-T))}|\phi_a\rangle.$$

We defined $\phi_i(\vec{x}) \coloneqq \phi(\vec{x}, x_i^0)$ to be an interaction picture field at some fixed time x_i^0, thus $\phi_i(\vec{x})$ is obviously time-independent. We therefore can define a Schrödinger operator $\phi_S(\vec{x})$ by the property $\phi_S(\vec{x}_1)|\phi_1\rangle = \phi_1(\vec{x}_1)|\phi_1\rangle$. That is, ϕ_S has nothing to do with ϕ_1, but is defined to *act* in the same way on $|\phi_1\rangle$. This yields

$$\begin{aligned}
&\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2) \exp\left(i \int d^4x\, \mathcal{L}\right) \\
&= \lim_{T\to\infty} \int \mathcal{D}\phi_1\, \mathcal{D}\phi_2\, \langle \phi_b | e^{-iH(T-x_2^0)} \phi_S(\vec{x}_2) | \phi_2 \rangle \langle \phi_2 | e^{-iH(x_2^0 - x_1^0)} \phi_S(\vec{x}_1) | \phi_1 \rangle \langle \phi_1 | e^{-iH(x_1^0-(-T))} | \phi_a \rangle \\
&= \lim_{T\to\infty} \langle \phi_b | e^{-iH(T-x_2^0)} \phi_S(\vec{x}_2) e^{-iH(x_2^0-x_1^0)} \phi_S(\vec{x}_1) e^{-iH(x_1^0-(-T))} | \phi_a \rangle \\
&= \lim_{T\to\infty} \langle \phi_b | e^{-iHT} \phi_H(x_2)\phi_H(x_1) e^{-iHT} | \phi_a \rangle
\end{aligned}$$

where we used $\mathbb{I} = \int \mathcal{D}\phi_1\, |\phi_1\rangle\langle\phi_2|$. We considered the case $x_1^0 < x_2^0$; for $x_2^0 < x_2^0$, the order would simply be interchanged and we can use the time-ordering operator to denote the general case as

$$\begin{aligned}
&\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2) \exp\left(i \int d^4x\, \mathcal{L}\right) = \lim_{T\to\infty} \langle \phi_b | e^{-iHT}\, \mathcal{T}\big(\phi_H(x_2)\phi_H(x_1)\big)\, e^{-iHT} | \phi_a \rangle \\
&\qquad = \langle \phi_a | \Omega \rangle \langle \Omega | \mathcal{T}\big(\phi_H(x_2)\phi_H(x_1)\big) | \Omega \rangle \langle \Omega | \phi_a \rangle.
\end{aligned}$$

In the last step we used

$$\begin{aligned}
\lim_{T\to\infty} e^{-iHT}|\phi_a\rangle &= \lim_{T\to\infty} \sum_n e^{-iHT}|n\rangle\langle n|\phi_a\rangle = \lim_{T\to\infty} \sum_n e^{-iE_nT}|n\rangle\langle n|\phi_a\rangle \\
&= |\Omega\rangle\langle\Omega|\phi_a\rangle + \lim_{T\to\infty} \sum_{n\neq\Omega} e^{-iE_nT}|n\rangle\langle n|\phi_a\rangle = |\Omega\rangle\langle\Omega|\phi_a\rangle,
\end{aligned}$$

where the term with the sum vanishes because of the Riemann-Lebesgue lemma, see (>7.9.1). Also, had to assume, that $|\phi_a\rangle$ has some overlap with $|\Omega\rangle$, such that $\langle\Omega|\phi_a\rangle \neq 0$. Note that for a free theory, we can assume that $|\phi_a\rangle$ has some overlap with the free vacuum $|0\rangle$ and simply replace $|\Omega\rangle$ by $|0\rangle$.

The denominator of the formula we want to proof gives simply

$$\int \mathcal{D}\phi \exp\left(i \int d^4x\, \mathcal{L}\right) = \lim_{T\to\infty} \langle \phi_b | e^{-iH(T-(-T))} | \phi_a \rangle = \langle \phi_a | \Omega \rangle \underbrace{\langle \Omega | \Omega \rangle}_{=1} \langle \Omega | \phi_a \rangle$$

and the proof is complete:

$$\begin{aligned}
&\frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2) \exp(i \int d^4x\, \mathcal{L})}{\int \mathcal{D}\phi\, \exp(i \int d^4x\, \mathcal{L})} = \frac{\langle \phi_a | \Omega \rangle \langle \Omega | \mathcal{T}\big(\phi_H(x_2)\phi_H(x_1)\big) | \Omega \rangle \langle \Omega | \phi_a \rangle}{\langle \phi_a | \Omega \rangle \langle \Omega | \phi_a \rangle} \\
&\qquad = \langle \Omega | \mathcal{T}\big(\phi_H(x_2)\phi_H(x_1)\big) | \Omega \rangle.
\end{aligned}$$

It is easily generalized to an arbitrary number of fields $\phi(x_1)\phi(x_2)\cdots\phi(x_n)$: One just needs introduce and integrate over n constraint fields $\phi_1, \phi_2, \dots, \phi_n$ and cut the main integral n times, yielding $n+1$ factors of transition amplitudes.

15.2.3 Two Point Function with Generating Functional

Using the chain rule for the functional derivative, we find

$$\begin{aligned}
-i\frac{\delta}{\delta J(z)} Z[J] &= -i\frac{\delta}{\delta J(z)} \int \mathcal{D}\phi \exp\left(i \int d^4x\, \big(\mathcal{L} + J(x)\phi(x)\big)\right) \\
&= \int \mathcal{D}\phi\, \phi(z) \exp\left(i \int d^4x\, \big(\mathcal{L} + J(x)\phi(x)\big)\right).
\end{aligned}$$

Thus,

$$\frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)Z[J]\bigg|_{J=0}$$
$$=\frac{1}{Z[0]}\int\mathcal{D}\phi\;\phi(x_1)\phi(x_2)\exp\left(i\int d^4x\,(\mathcal{L}+J(x)\phi(x))\right)\bigg|_{J=0}$$
$$=\frac{\int\mathcal{D}\phi\;\phi(x_1)\phi(x_2)\exp(i\int d^4x\,\mathcal{L})}{\int\mathcal{D}\phi\exp(i\int d^4x\,\mathcal{L})}=\langle 0|\mathcal{T}\phi(x_1)\phi(x_2)|0\rangle.$$

15.2.4 Generating Functional of the Free Klein-Gordon Field

In free Klein-Gordon theory, the Lagrangian reads $\mathcal{L}_0=(\partial^\mu\phi)^2/2-m^2\phi^2/2$. Integration by parts yields for the exponent of the generating functional

$$i\int d^4x\,(\mathcal{L}_0+J\phi)=i\int d^4x\left(\frac{1}{2}\left((\partial^\mu\phi)(\partial_\mu\phi)-m^2\phi^2\right)+J\phi\right)$$
$$=i\int d^4x\,\left(\frac{1}{2}(-\phi\Box\phi-m^2\phi^2)+J\phi\right)=-i\int d^4x\,\left(\frac{1}{2}\phi(\Box+m^2-i\epsilon)\phi-J\phi\right).$$

The functional integral $\mathcal{D}\phi$ over the exponential function with this exponent is basically a Gaussian integral (due to the structure[1] $\phi A\phi+b\phi$). Since it is purely imaginary for real fields ϕ, we had to include an $-i\epsilon$ in the ϕ^2-term to ensure convergence (see also footnote on page 129). We can complete the square by introducing the substitution

$$\phi=\phi'+\underbrace{\int d^4y\,iD_F(x-y)\,J(y)}_{=:I},$$

such that ϕ' is our new integration variable for the functional integral (obviously, $\mathcal{D}\phi=\mathcal{D}\phi'$). When plug in $\phi=\phi'+I$ and abbreviate $G\coloneqq\Box+m^2-i\epsilon$ we find

$$i\int d^4x\,(\mathcal{L}_0+J\phi)=-i\int d^4x\,\left(\frac{1}{2}(\phi'+I)G(\phi'+I)-J(\phi'+I)\right)$$
$$=-i\int d^4x\,\left(\frac{1}{2}(\phi'G\phi'+\phi'GI+IG\phi'+IGI)-J(\phi'+I)\right).$$

We can now make use of the fact that iD_F is the Greens function of the Klein-Gordon operator G, that is $GiD_F(x-y)=\delta(x-y)$, as we found in section 4.8. Also, since it contains two derivatives $\Box=\partial^2$,

[1] In analogy, for a matrix $A\in\mathbb{R}^{n\times n}$ we have
$$\int(\Pi_i\xi_i)\exp\left(-\xi_iA_{ij}\xi_j\right)=\sqrt{\frac{\pi^n}{\det A}},$$
the derivation of which is based on Gaussian integrals: We can substitute $\xi_i=O_{ij}x_j$ where O_{ij} is the orthogonal matrix of eigenvectors of A, such that
$$\int(\Pi_i\xi_i)\exp\left(-\xi_iA_{ij}\xi_j\right)=\int(\Pi_ix_i)\exp\left(-x_kO^T_{ik}A_{ij}O_{jl}x_l\right)=\int(\Pi_ix_i)\exp(-\Sigma_kx_ka_k\delta_{kl}x_l)$$
$$=\int(\Pi_ix_i)\exp(-a_ix_i^2)=\prod_i\sqrt{\frac{\pi}{a_i}}=\sqrt{\frac{\pi^n}{\det(O^TAO)}}=\sqrt{\frac{\pi^n}{\det A}}.$$
a_i are the Eigenvalues of A. The Jacobi matrix is in this case the orthogonal O, thus the Jacobi determinant is 1.
Obviously, this holds for all matrices, whose eigenvalues have a non-vanishing positive real part (see footnote on page 129). In the case that A has purely real (and positive) eigenvalues, the integral with an additional i in the exponent will give, adding an infinitesimal $i\epsilon$ (see, again, footnote on page 129),
$$\int(\Pi_i\xi_i)\exp\left(-i\xi_iA_{ij}\xi_j\right)=\int(\Pi_i\xi_i)\exp\left(-i\xi_i\left(A_{ij}-i\epsilon\right)\xi_j\right)=\cdots=\prod_i\sqrt{\frac{-i\pi}{a_i-i\epsilon}}=\sqrt{\frac{(-i\pi)^n}{\det A-i\epsilon}}.$$

we can always integrate by parts twice to transfer G to act on the left-hand function underneath an integral. So let's evaluate the terms individually:

$$\phi' G I = \phi'(x) G \int d^4y\, iD_F(x-y)\, J(y) = \phi'(x) \int d^4y\, \delta(x-y)\, J(y) = \phi'(x) J(x),$$

$$\begin{aligned} I G \phi' &= \int d^4y\, iD_F(x-y)\, J(y) G \phi'(x) = \int d^4y\, G iD_F(x-y)\, J(y)\, \phi'(x) \\ &= \int d^4y\, \delta(x-y)\, J(y)\, \phi'(x) = J(x)\phi'(x) = \phi' G I, \end{aligned}$$

$$\begin{aligned} I G I &= \int d^4y\, iD_F(x-y)\, J(y)\, G \int d^4y'\, iD_F(x-y')\, J(y') \\ &= \int d^4y\, iD_F(x-y)\, J(y) \int d^4y'\, \delta(x-y')\, J(y') = \int d^4y\, J(x)\, iD_F(x-y)\, J(y) = JI. \end{aligned}$$

Thus, we arrive at

$$\begin{aligned} i \int d^4x\, (\mathcal{L}_0 + J\phi) &= -i \int d^4x\, \left(\frac{1}{2}(\phi' G \phi' + 2\phi' J + JI) - J\phi' - JI\right) \\ &= -i \int d^4x\, \left(\frac{1}{2}\phi' G \phi' - \frac{1}{2} JI\right) = -i \int d^4x\, \left(\mathcal{L}_0(\phi') - \frac{1}{2}\int d^4y\, J(x)\, iD_F(x-y)\, J(y)\right). \end{aligned}$$

In the functional integral, we can simply change $\mathcal{D}\phi \to \mathcal{D}\phi'$, as the Jacobian of a simply shift is 1. Thus, the generating functional of the free Klein-Gordon theory reads

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \exp\left(i \int d^4x\, (\mathcal{L}_0 + J(x)\phi(x))\right) \\ &= \int \mathcal{D}\phi' \exp\left(-i \int d^4x\, \left(\mathcal{L}_0(\phi') - \frac{1}{2}\int d^4y\, J(x)\, iD_F(x-y)\, J(y)\right)\right) \\ &= \underbrace{\int \mathcal{D}\phi' \exp\left(-i \int d^4x\, \mathcal{L}_0(\phi')\right)}_{=Z[0]} \exp\left(\frac{i}{2}\int d^4x\, d^4y\, J(x)\, iD_F(x-y)\, J(y)\right). \end{aligned}$$

Note, that the second exponential function is independent of ϕ' and can therefore be pulled out of the $\mathcal{D}\phi'$-integral.

15.2.5 Evaluating 2-Point Function with Functional Integrals

We already know from section 4.8 that for the real scalar field (which is Hermitian, $\phi^\dagger = \phi$) we have

$$\langle 0|\mathcal{T}\phi(x_1)\phi(x_2)|0\rangle = D_F(x_1 - x_2) = \int d^4\bar{p} \frac{i\, e^{-ip\cdot(x_1-x_2)}}{p^2 - m^2 + i\epsilon}.$$

We now want to find this result using the functional integral formulation. Using our result of (>15.2.3) and the explicit form of the generating function for the free Klein-Gordon field from (>15.2.4), we find

$$\begin{aligned} \langle 0|\mathcal{T}\phi(x_1)\phi(x_2)|0\rangle &= \left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\frac{Z[J]}{Z[0]}\bigg|_{J=0} \\ &= \left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)\exp\left(\frac{i}{2}\int d^4x\, d^4y\, J(x)\, iD_F(x-y)\, J(y)\right)\bigg|_{J=0} \\ &= \left(-i\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{2}\frac{\delta}{\delta J(x_2)}\int d^4x\, d^4y\, J(x)\, iD_F(x-y)\, J(y)\right)\frac{Z[J]}{Z[0]}\bigg|_{J=0} \\ &= \left(-i\frac{\delta}{\delta J(x_1)}\right)\frac{1}{2}\left(\int d^4y\, iD_F(x_2-y)\, J(y) + \int d^4x\, J(x)\, iD_F(x-x_2)\right)\frac{Z[J]}{Z[0]}\bigg|_{J=0} \\ &= \left(-i\frac{\delta}{\delta J(x_1)}\right)\int d^4x\, J(x)\, iD_F(x-x_2)\frac{Z[J]}{Z[0]}\bigg|_{J=0}. \end{aligned}$$

Since $D_F(x-y) = D_F(y-x)$, the two terms of the product rule are identical. Now, the derivative $\delta/\delta J(x_1)$ acts via product rule on the J in the x-integral and also on $Z[J]$. However, the term where it acts on $Z[J]$ will vanish when J is set to zero. Thus, only the following term remains:

$$\langle 0|\mathcal{T}\phi(x_1)\phi(x_2)|0\rangle = -i\int d^4x\, \delta(x-x_1)\, iD_F(x-x_2)\frac{Z[J]}{Z[0]}\bigg|_{J=0} = D_F(x_1-x_2).$$

15.2.6 Evaluating 4-Point Function with Functional Integrals

For the calculation of the 4-point function, we will use the following abbreviations:

$$\phi_i \coloneqq \phi(x_i), \qquad J_i \coloneqq J(x_i), \qquad J_x \coloneqq J(x), \qquad D_{xi} \coloneqq D(x-x_i).$$

Also, integration over double indices x, y, z is implicit. Then we find

$$\begin{aligned}
\langle 0|\mathcal{T}\phi_1\phi_2\phi_3\phi_4|0\rangle &= (-i)^4\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4}\frac{Z[J]}{Z[0]}\bigg|_{J=0} = (-i)^4\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4}\exp\left(\frac{i}{2}J_x\, iD_{xy}\, J_y\right)\bigg|_{J=0} \\
&= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\left(-\frac{1}{2}\left(D_{4y}J_y + J_xD_{x4}\right)\right)\exp\left(-\frac{1}{2}J_x\, D_{xy}\, J_y\right)\bigg|_{J=0} \\
&= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}(-J_xD_{x4})\exp\left(-\frac{1}{2}J_x\, D_{xy}\, J_y\right)\bigg|_{J=0} \\
&= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\left(-D_{34} + J_xD_{x4}J_yD_{y3}\right)\exp\left(-\frac{1}{2}J_x\, D_{xy}\, J_y\right)\bigg|_{J=0} \\
&= \frac{\delta}{\delta J_1}\left(D_{34}J_xD_{x2} + D_{24}J_yD_{y3} + J_xD_{x4}D_{23} - J_xD_{x4}J_yD_{y3}J_zD_{z2}\right)\exp\left(-\frac{1}{2}J_x\, D_{xy}\, J_y\right)\bigg|_{J=0} \\
&= D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23}.
\end{aligned}$$

All of the many more terms created by the derivative $\delta/\delta J_1$ in the last step vanished as we set $J=0$.

15.3 Quantization of the Electromagnetic Field

15.3.1 The Problem of Gauge Invariance

Recall the formula from section 14.2, which connected the n-point function with a fraction of functional integrals. The denominator of this fraction reads, if we exchange the scalar field ϕ by the vector field A,

$$\int \mathcal{D}A \exp\left(i\int d^4x\, \mathcal{L}\right),$$

where $\mathcal{D}A \coloneqq \mathcal{D}A^0\mathcal{D}A^1\mathcal{D}A^2\mathcal{D}A^3$ is the functional integral over all components of the vector field. As known from section 3.6, the Lagrangian of the *free* electromagnetic field reads $\mathcal{L} = -F_{\mu\nu}F^{\mu\nu}/4$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using integration by parts at the third equal sign, we can write the exponent as

$$\begin{aligned}
i\int d^4x\, \mathcal{L} &= -\frac{i}{4}\int d^4x\, \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{i}{2}\int d^4x\, \left((\partial^\mu A^\nu)(\partial_\mu A_\nu) - (\partial_\nu A_\mu)(\partial^\mu A^\nu)\right) = \frac{i}{2}\int d^4x\, \left(A^\nu\Box A_\nu - A_\mu\partial_\nu\partial^\mu A^\nu\right) \\
&= -i\int d^4x\, \frac{1}{2}A_\mu(-\eta^{\mu\nu}\Box + \partial^\nu\partial^\mu - i\epsilon)A_\nu.
\end{aligned}$$

Note, that $(\eta^{\mu\nu}\Box - \partial^\nu\partial^\mu)A_\nu = \partial_\nu F^{\nu\mu}$ and that we found $\partial_\nu F^{\nu\mu} = 0$ to be the free equations of motions for the electromagnetic field in section 3.6. In the same way as for the real scalar field in (>15.2.4), we introduced $-i\epsilon$ to ensure convergence of the integral.

When we did this calculation for the scalar field in (>15.2.4), we introduced a shift of the integration variable of the functional integral (that is, the field) and the shift contained the Feynman propagator. This was helpful due to the fact that the Feynman propagator is the Greens function of the Klein-Gordon operator. By analogy, it would be helpful to shift the field A^μ by the Greens function of the operator $\eta^{\mu\nu}\Box - \partial^\nu\partial^\mu$, that is the function $D_F^{\nu\sigma}(x-y)$ defined by

$$\left(-\eta_{\mu\nu}\Box + \partial_\nu\partial_\mu - i\epsilon\right)i\widehat{D}_F^{\nu\sigma}(x-y) = \delta_\mu^\sigma\ \delta(x-y).$$

Unfortunately, such an equation has no solution $\widehat{D}_F^{\nu\sigma}(x-y)$. To see this, we can consider the Fourier transform of this equation,

$$\left(-\eta_{\mu\nu}k^2 + k_\mu k_\nu - i\epsilon\right)i\widehat{D}_F^{\nu\sigma}(k) = \delta_\mu^\sigma.$$

If we take $\eta^{\mu\nu}k^2 - k^\mu k^\nu$ to be the 4×4 matrix and consider this equation to be a matrix equation, we could find $\widehat{D}_F^{\nu\sigma}$ by multiplying the equation with the invers matrix of $\eta^{\mu\nu}k^2 - k^\mu k^\nu$. However, the determinant of the matrix $\eta^{\mu\nu}k^2 - k^\mu k^\nu$ turns out to be zero, such that no invers matrix exists.

This difficulty is due to gauge invariance. $F^{\mu\nu}$ and thus $\mathcal{L}$ is invariant under a general gauge transformation of the form[1]

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x).$$

This gauge invariance yields that the functional integral is badly defined, since we redundantly integrate over a continuous infinity of physically equivalent field configurations.

15.3.2 Faddeev-Popov Procedure

To avoid the problem just discussed in the end of (>15.3.1), we would like to isolate the interesting part of the functional integral, which counts each physical configuration only once. Let $G(A) = 0$ be our gauge condition; that is for the Lorentz gauge we choose the function $G(A)$ to be $G(A) = \partial_\mu A^\mu$. We want to include a "$\delta$-function" $\delta[G(A)]$ into our integral to restrain the functional integral to the gauge $G(A) = 0$. To do so legally, we can introduce the following form of a 1:[2]

$$1 = \int \mathcal{D}\alpha\ \delta\big(G(A^\alpha)\big)\left|\det\frac{\delta G(A^\alpha)}{\delta\alpha}\right|, \qquad \text{where} \qquad A_\mu^\alpha \coloneqq A_\mu + \partial_\mu\alpha.$$

For the Lorentz gauge, we have

$$\frac{\delta G(A^\alpha)}{\delta\alpha} = \frac{\delta}{\delta\alpha}\left(\partial^\mu A_\mu^\alpha\right) = \frac{\delta}{\delta\alpha}\left(\partial^\mu A_\mu + \Box\alpha\right) = \Box$$

and $\det\Box$ is a functional determinant (since operators are analogue to matrices, one can also define determinants of them). For the present discussion, the definition of a functional determinant is irrelevant. What is important, is that it is independent of A, so we can pull it in front of the $\mathcal{D}A$-integral. Thus, plugging in the 1 in the form given above, we can write

$$\int \mathcal{D}A \exp\left(i\int d^4x\ \mathcal{L}\right) = \left|\det\frac{\delta G(A^\alpha)}{\delta\alpha}\right| \int \mathcal{D}A\ \mathcal{D}\alpha\ \delta\big(G(A^\alpha)\big)\exp\left(i\int d^4x\ \mathcal{L}\right).$$

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[1] Let's quickly check this statement:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \to \partial^\mu(A^\nu + \partial^\nu\alpha) - \partial^\nu(A^\mu + \partial^\mu\alpha) = F^{\mu\nu}.$$

[2] This identity is the analogue of

$$\int dx\ \delta\big(g(x)\big)\,|g'(x)| = \int dx\,\frac{\delta(x-x_0)}{|g'(x_0)|}\,|g'(x)| = \int dx\ \delta(x-x_0) = 1$$

$$\text{or} \qquad \int_{\mathbb{R}^n} d^n x\ \delta\big(\vec{g}(\vec{x})\big)\,|\det\vec{g}'(\vec{x})| = \int_{g(\mathbb{R}^n)} d^n x\ \delta(\vec{x}) = 1,$$

where $\det\vec{g}'(\vec{x})$ is the determinant of the matrix with elements $\partial g_i/\partial x_j$.

We now shift $A \to A - \partial\alpha$, such that the A^α in the "δ-function" becomes simply A:

$$\begin{aligned}\int \mathcal{D}A \exp\left(i \int d^4x\, \mathcal{L}\right) &= \left|\det \frac{\delta G(A^\alpha)}{\delta\alpha}\right| \int \mathcal{D}(A - \partial\alpha)\, \mathcal{D}\alpha\, \delta\big(G(A)\big) \exp\left(i \int d^4x\, \mathcal{L}(A - \partial\alpha)\right) \\ &= \left|\det \frac{\delta G(A^\alpha)}{\delta\alpha}\right| \int \mathcal{D}A\, \mathcal{D}\alpha\, \delta\big(G(A)\big) \exp\left(i \int d^4x\, \mathcal{L}(A)\right).\end{aligned}$$

Since this is a pure shift, we had $\mathcal{D}(A - \partial\alpha) = \mathcal{D}A$ and by gauge invariance we had $\mathcal{L}(A - \partial\alpha) = \mathcal{L}(A)$.

To move on, we need to choose a gauge fixing. We will choose the somewhat generalized Lorentz gauge

$$G(A) = \partial^\mu A_\mu(x) - \omega(x)$$

with an arbitrary scalar function ω. This gauge still has the functional determinant $\det \delta G(A^\alpha)/\delta\alpha = \det \Box$:

$$\int \mathcal{D}A \exp\left(i \int d^4x\, \mathcal{L}\right) = |\det \Box| \int \mathcal{D}A\, \mathcal{D}\alpha\, \delta\left(\partial^\mu A_\mu(x) - \omega(x)\right) \exp\left(i \int d^4x\, \mathcal{L}(A)\right).$$

ω is arbitrary, so this equation holds for any ω. Hence, we can replace the right-hand side with any (properly normalized) linear combination each term involving a different function ω_i:

$$\int \mathcal{D}A \exp\left(i \int d^4x\, \mathcal{L}\right) = |\det \Box| \sum_i C_i(\omega_i) \int \mathcal{D}A\, \mathcal{D}\alpha\, \delta\left(\partial^\mu A_\mu(x) - \omega_i(x)\right) \exp\left(i \int d^4x\, \mathcal{L}(A)\right),$$

where the coefficients may even depend on ω, as long as they are properly normalized. Instead of this sum, we can also integrate over ω and we choose the weights $C_i(\omega_i)$ to be a Gaussian function together with some normalization factors N:

$$\begin{aligned}&\int \mathcal{D}A \exp\left(i \int d^4x\, \mathcal{L}\right) \\ &\quad= |\det \Box|\, N(\xi) \int \mathcal{D}\omega \exp\left(-i \int d^4x \frac{\omega^2}{2\xi}\right) \int \mathcal{D}A\, \mathcal{D}\alpha\, \delta\left(\partial^\mu A_\mu(x) - \omega(x)\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \exp\left(i \int d^4x\, \mathcal{L}(A)\right) \\ &\quad= |\det \Box|\, N(\xi) \int \mathcal{D}A\, \mathcal{D}\alpha\, \exp\left(-i \int d^4x \frac{1}{2\xi}\left(\partial^\mu A_\mu\right)^2\right) \exp\left(i \int d^4x\, \mathcal{L}(A)\right).\end{aligned}$$

In the last step, we have used the "δ-function" to evaluate the $\mathcal{D}\omega$-integral. Effectively, we have added a new term $-(\partial^\mu A_\mu)^2/2\xi$ to the Lagrangian. ξ is completely arbitrary, as long as the normalization $N(\xi)$ as accordingly chosen.

Consider

$$\langle\Omega|\mathcal{T}\, O(A)|\Omega\rangle = \frac{\int \mathcal{D}A\, O(A) \exp(i \int d^4x\, \mathcal{L})}{\int \mathcal{D}A \exp(i \int d^4x\, \mathcal{L})}.$$

Here, we just wrote a general operator $O(A)$ instead of the product $A(x_1)A(x_2) \cdots A(x_n)$ that appears in a n-point function. We have already taken care of the denominator. As long as $O(A)$ is gauge invariant,[1] we can do exactly the same steps also for the numerator and find

$$\langle\Omega|\mathcal{T}\, O(A)|\Omega\rangle = \frac{\int \mathcal{D}A\, O(A) \exp\left(i \int d^4x\, \left(\mathcal{L} - \frac{1}{2\xi}\left(\partial^\mu A_\mu\right)^2\right)\right)}{\int \mathcal{D}A \exp\left(i \int d^4x\, \left(\mathcal{L} - \frac{1}{2\xi}\left(\partial^\mu A_\mu\right)^2\right)\right)}.$$

[1] This is needed in the one step where we also exploited the gauge invariance of the Lagrangian.

The factors $|\det\Box| N(\xi) \int \mathcal{D}\alpha$ cancel, as they occur in the same way in the numerator and denominator.

Let's now plug in the Lagrangian as in (>15.3.1) and add the additional gauge term:

$$i\int d^4x \left(\mathcal{L} - \frac{1}{2}\xi^{-1}(\partial^\mu A_\mu)(\partial^\nu A_\nu)\right) = -i\int d^4x \frac{1}{2} A_\mu(-\eta^{\mu\nu}\Box + (1-\xi^{-1})\partial^\nu\partial^\mu - i\epsilon)A_\nu,$$

where we performed integration by parts (which yields the minus sign in front of ξ^{-1}). In (>15.3.1) we tried, but were unable to, find the Greens function of the operator $-\eta^{\mu\nu}\Box + (1-\xi^{-1})\partial^\nu\partial^\mu - i\epsilon$ with $\xi^{-1} = 0$. Let's see, if we are more successful with the additional ξ-term. In Fourier space the relevant equation reads

$$\left(\eta_{\mu\nu}k^2 - (1-\xi^{-1})k_\mu k_\nu - i\epsilon\right) i\widehat{D}_F^{\nu\sigma}(k) = \delta_\mu^\sigma,$$

which is indeed solved by[1]

$$\widehat{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right).$$

15.4 Graßmann Numbers

15.4.1 Integrals of Graßmann Numbers

For our purposes we do not need definite integrals (that is, integrals with borders), but only the analogue of $\int_{-\infty}^{\infty} dx$, that is integrals over all possible values of x. As we already learned, any function of a Graßmann variable can be expanded as a linear function, $f(\theta) = A + B\theta$, since all the higher terms of a Taylor series contain at least one factor $\theta^2 = 0$. Imposing usual integration rules, we can write

$$\int d\theta\, f(\theta) = \int d\theta\, (A + B\theta) = A\int d\theta + B\int d\theta\, \theta.$$

When working with functional integrals, we often shifted the integration variable without changing the integral. We also want this feature for integrals of Graßmann numbers, that is we demand that the integral does not change under a shift $\theta \to \theta + \eta$:

$$\begin{aligned} A\int d\theta + B\int d\theta\,\theta &= \int d\theta\, f(\theta) \stackrel{!}{=} \int d\theta\, f(\theta+\eta) = \int d\theta\,\big(A + B(\theta+\eta)\big) \\ &= \int d\theta\,\big((A + B\eta) + B\theta\big) = (A - B\eta)\int d\theta + B\int d\theta\,\theta \qquad \Leftrightarrow \qquad -B\eta\int d\theta \stackrel{!}{=} 0 \end{aligned}$$

(note that we commuted $d\theta\,\eta = -\eta\, d\theta$). Since this should hold for any complex B and any Graßmann number η, we have $\int d\theta = 0$. The integral $\int d\theta\,\theta$ is in principle an arbitrary constant, conventionally taken to be 1. Thus,

$$\int d\theta\, f(\theta) = \int d\theta\,(A + B\theta) = B.$$

15.4.2 Normal Gaussian Integrals with Graßmann Numbers

When we want to evaluate the integral over $e^{-\theta^* a\theta}$, we expand the exponential function as

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[1] *When I plugged this solution into the Fourier transformed equation, everything cancels nicely and what remains is*

$$\delta_\mu^\sigma \frac{k + \sigma i\epsilon}{k + i\epsilon},$$

where σ is a bunch of terms; in principle, σ is irrelevant, since we can always rename $\sigma\epsilon \to \tilde{\epsilon} \to \epsilon$. However, σ can be negative (depending, for example, on the choice of ξ). And then it's probably not so easy anymore to say that the result is just δ_μ^σ. The point is: I cannot justify the sign of the $i\epsilon$ in the denominator of the Feynman propagator.

$$e^{-\theta^* a\theta} = 1 - \theta^* a\theta + \frac{1}{2}\underbrace{(\theta^* a\theta)^2}_{0} + \underbrace{\cdots}_{=0} = 1 - \theta^* a\theta,$$

where all higher order terms vanish due to $\theta^2 = \theta^{*2} = 0$. Using our simple rules of integration, we find

$$\int d\theta^* d\theta\, e^{-\theta^* a\theta} = \int d\theta^* d\theta\, (1 - \theta^* a\theta) = \int d\theta^* d\theta\, \theta\theta^* a = a,$$

$$\int d\theta^* d\theta\, \theta\theta^*\, e^{-\theta^* a\theta} = \int d\theta^* d\theta\, \theta\theta^*\, (1 - \theta^* a\theta) = \int d\theta^* d\theta\, \theta\theta^* = 1.$$

For ordinary complex numbers we would get, using $z = x + iy$,

$$\int dz^* dz\, e^{-z^* az} = \int dx\, dy\, e^{-a(x^2+y^2)} = \frac{\pi}{a},$$

$$\int dz^* dz\, zz^*\, e^{-z^* az} = \int dx\, dy\, (x^2 + y^2)\, e^{-a(x^2+y^2)} = 2\int dx\, x^2 e^{-ax^2} \int dy\, e^{-ay^2}$$
$$= 2\frac{1}{2a}\sqrt{\frac{\pi}{a}}\sqrt{\frac{\pi}{a}} = \frac{1}{a}\frac{\pi}{a}.$$

For Graßmann numbers as well as for ordinary numbers, the second integral adds an factor $1/a$ to the result of the first integral.

15.4.3 Multidimensional Gaussian Integrals with Graßmann Numbers

We want to evaluate

$$\int (\Pi_i d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij}\theta_j}.$$

We already evaluated this integral for normal (and real) numbers in the footnote on page 136. Now that we have complex numbers, we need that the matrix U of eigenvectors of A is unitary (instead of orthogonal).[1] Plugging in $\theta_i = U_{ij}\theta_j'$, we find

$$\int (\Pi_i d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij}\theta_j} = \int (\Pi_i d\theta_i'^* d\theta_i') e^{-\theta_j'^* U_{ji}^\dagger A_{ij} U_{jk}\theta_k'} = \int (\Pi_i d\theta_i'^* d\theta_i') e^{-\sum_{jk}\theta_j'^* a_j \delta_{jk}\theta_k'}$$
$$= \int (\Pi_i d\theta_i'^* d\theta_i') e^{-\sum_k \theta_k'^* a_k \theta_k'} = \int (\Pi_i d\theta_i'^* d\theta_i') \prod_k e^{-\theta_k'^* a_k \theta_k'}$$
$$= \int (\Pi_i d\theta_i'^* d\theta_i') \prod_k (1 - a_k \theta_k'^* \theta_k') = \int (\Pi_i d\theta_i'^* d\theta_i') \prod_k (1 + a_k \theta_k' \theta_k'^*).$$

In the last step, we just reversed $\theta_k'^*\theta_k' = -\theta_k'\theta_k'^*$. Of this product over k, all terms vanish after integration except the single factor containing each Graßmann variable once:

[1] Before we move on, we should check if an integral over complex *Graßmann* numbers is invariant under unitary transformations. If $\theta_i = U_{ij}\theta_j'$, then

$$\Pi_{i=1}^n \theta_i = \theta_1 \cdots \theta_n = \frac{1}{n!}\epsilon^{ij\ldots k}\, \theta_i\theta_j \cdots \theta_k = \frac{1}{n!}\epsilon^{ij\ldots k}\, U_{ii'}\theta_{i'}' U_{jj'}\theta_{j'}' \cdots U_{kk'}\theta_{k'}'$$
$$= \frac{1}{n!}\epsilon^{ij\ldots k}\, U_{ii'}U_{jj'}U_{kk'}\, \theta_{i'}'\theta_{j'}' \cdots \theta_{k'}' = \frac{1}{n!}\epsilon^{ij\ldots k}\, U_{ii'}U_{jj'} \cdots U_{kk'}\, \epsilon^{i'j'\ldots k'}\, \Pi_l \theta_l' = \det U\ \Pi_i\theta_i'.$$

In a general integral $\int (\Pi_i\, d\theta_i^*\, d\theta_i)\, f(\{\theta_i\},\{\theta_i^*\})$, the only term of f that survives has exactly one factor of each θ_i and each θ_i^*; thus f is effectively (under the integral) proportional to $(\Pi_i\theta_i)(\Pi_j\theta_j^*) = \det U \det U^* (\Pi_i\theta_i')(\Pi_j\theta_j'^*)$. Since U is unitary, it holds $\det U \det U^* = |\det U| = 1$. Thus, the integral is unchanged under a unitary transformation.

$$\int (\Pi_i d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij}\theta_j} = \int (\Pi_i d\theta_i'^* d\theta_i') \prod_k a_k \theta_k'^* \theta_k' = (\Pi_k a_k) \int \Pi_i d\theta_i'^* d\theta_i'\, \theta_i'^* \theta_i' = \Pi_k a_k$$
$$= \det U^\dagger A U = \det A.$$

Now, let's tackle the second integral, with two Graßmann numbers $\theta_n \theta_m^*$ in front of the exponential function. This one is slightly more complicated (the single steps a explained below in detail):

$$\int (\Pi_i d\theta_i^* d\theta_i)\, \theta_n \theta_m^*\, e^{-\theta_i^* A_{ij}\theta_j} = \int (\Pi_i d\theta_i'^* d\theta_i')\, U_{nn'} U^\dagger_{m'm} \theta'_{n'} \theta'^*_{m'} \prod_k (1 + a_k \theta_k' \theta_k'^*)$$
$$= \sum_{n'} \int (\Pi_i d\theta_i'^* d\theta_i')\, U_{nn'} U^\dagger_{n'm} \theta'_{n'} \theta'^*_{n'} \prod_{k \neq n'} a_k \theta_k' \theta_k'^*$$
$$= \sum_{n'} \left(\prod_{k \neq n'} a_k \right) U_{nn'} U^\dagger_{n'm} \underbrace{\int (\Pi_i d\theta_i'^* d\theta_i')\, \theta'_{n'} \theta'^*_{n'} \prod_{k \neq n'} \theta_k' \theta_k'^*}_{=1} = \det A \sum_{n'} \frac{1}{a_{n'}} U_{nn'} U^\dagger_{n'm}$$
$$= \det A \sum_{n'} U_{nn'} D^{-1}_{n'n'} U^\dagger_{n'm} = \det A \left(U D^{-1} U^\dagger \right)_{nm} = A^{-1}_{nm} \det A.$$

This is what happens at the n-th equal sign:

1. As we already did for the first integral, we substituted $\theta_i = U_{ij}\theta_j'$ and wrote the exponential as $e^{-\theta_i^* A_{ij}\theta_j} = \prod_k (1 + a_k \theta_k' \theta_k'^*)$ (see the first integral at the beginning of the current section for details).
2. Of the large product over k only those terms survive the integration, in which each θ_i' and each θ_i' occurs (since $\int d\theta = 0$). Since we already have the two factors $\theta'_{n'}\theta'^*_{m'}$, they cannot appear in the product over k. However, since in the product over k the factors θ_k' and $\theta_k'^*$ occur always pairwise, n' and m' must be equal and it is exactly this index which must be missing in the product over k. Therefore, we set $m' = n'$ and since n' is no double index anymore, we write the sum $\Sigma_{n'}$ explicitly. Of the product over k we want to have the term with all the $\theta_k'^*\theta_k'$ except for $\theta'^*_{n'}\theta'_{n'}$.
3. We can pull out the constant matrix elements U_{ij} and all the factors a_k in front of the integral. What remains is just an integral over all Graßmann numbers appearing precisely once. According to our definition of integrals over Graßmann variables, this is 1.
4. We wrote $\Pi_{k \neq n'} a_k = a_{n'}^{-1} \Pi_k a_k$ and used $\Pi_k a_k = \det D = \det A$, where D is the diagonal matrix to A.
5. The invers of a diagonal matrix with entries a_n on the diagonal is also diagonal and has entries $1/a_n$ on its diagonal. Thus, since $D_{nn} = a_n$, we used $D^{-1}_{nn} = 1/a_n$.
6. Since D^{-1} is diagonal, it holds $\sum_i U_{ni} D^{-1}_{ii} U^\dagger_{im} = U_{ni} D^{-1}_{ij} U^\dagger_{jm} = \left(U D^{-1} U^\dagger\right)_{nm}$.
7. Here we simply used $D = U^\dagger A U \Leftrightarrow D^{-1} = U^\dagger A^{-1} U \Leftrightarrow U D^{-1} U^\dagger = A^{-1}$ (recall that U is unitary, that is $U^{-1} = U^\dagger$).

15.5 Quantization of Spinor Fields

15.5.1 Evaluate 2-Point Function with Functional Integrals

Using the general Gaussian integrals over Graßmann numbers from section 14.4,

$$\int (\Pi_i d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij}\theta_j} = \det A, \qquad \int (\Pi_i d\theta_i^* d\theta_i)\, \theta_n \theta_m^*\, e^{-\theta_i^* A_{ij}\theta_j} = A^{-1}_{nm} \det A,$$

we find that

$$\int \mathcal{D}\psi\, \mathcal{D}\bar\psi\, \exp\left(i \int d^4x\, \bar\psi (i\partial\!\!\!/ - m + i\epsilon)\psi \right) = \int \mathcal{D}\psi\, \mathcal{D}\bar\psi\, \exp\left(-\int d^4x\, \bar\psi (\partial\!\!\!/ + im + \epsilon)\psi \right)$$
$$= \det(\partial\!\!\!/ + im + \epsilon),$$

$$\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\exp\left(i\int d^4x\,\bar{\psi}(i\partial\!\!\!/ - m + i\epsilon)\psi\right)\psi(x_1)\bar{\psi}(x_2)$$
$$= \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\exp\left(-\int d^4x\,\bar{\psi}(\partial\!\!\!/ + im + \epsilon)\psi\right)\psi(x_1)\bar{\psi}(x_2)$$
$$= (\partial\!\!\!/ + im + \epsilon)^{-1}\det(\partial\!\!\!/ + im + \epsilon)$$

and hence

$$\widetilde{D}_F(x_1 - x_2) \coloneqq \langle 0|\mathcal{T}\psi(x_1)\bar{\psi}(x_2)|0\rangle = \frac{\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\exp(i\int d^4x\,\bar{\psi}(i\partial\!\!\!/ - m + i\epsilon)\psi)\,\psi(x_1)\bar{\psi}(x_2)}{\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\ \exp(i\int d^4x\,\bar{\psi}(i\partial\!\!\!/ - m + i\epsilon)\psi)}$$
$$= (\partial\!\!\!/ + im + \epsilon)^{-1} = \left(-i(i\partial\!\!\!/ - m + i\epsilon)\right)^{-1},$$

where $(\partial\!\!\!/ + im + \epsilon)^{-1}$ is the inverse operator of $\partial\!\!\!/ + im + \epsilon$.

An inverse operator A^{-1} is something like a Greens function G of the corresponding operator A. After all, $AA^{-1} = 1$ is not so different from $AG(x) = \delta(x)$. The precise relationship is

$$\int d^4y\,G(x-y)J(y) = A^{-1}J(x)$$

for an arbitrary function $J(x)$. This is obvious, when we apply A from the left and use $AG(x-y) = \delta(x-y)$. Using that for matrices as well as operators $(aA)^{-1} = a^{-1}A^{-1}$ for some constant a is true, we have found above that

$$\widetilde{D}_F(x_1 - x_2) = i(i\partial\!\!\!/ - m + i\epsilon)^{-1}.$$

This equality between a function $\widetilde{D}_F$ and an (inverse) operator is understood to be exactly in the sense given above, namely that for any function $J(x)$ it holds

$$\int d^4y\,\widetilde{D}_F(x-y)J(y) = i(i\partial\!\!\!/ - m + i\epsilon)^{-1}J(x).$$

Thus, it must also hold for the choice $J(x) = e^{-ik\cdot x}$. Let's plug this in, apply the inverse operator on it and write the equation in from of a Fourier transformation by multiplying $e^{ik\cdot x}$ on both sides:

$$\int d^4y\,\widetilde{D}_F(x-y)e^{-ik\cdot y} = \frac{i}{k\!\!\!/ - m + i\epsilon}e^{-ik\cdot x}$$
$$\Leftrightarrow \qquad \int d^4y\,\widetilde{D}_F(x-y)e^{ik\cdot(x-y)} = \frac{i}{k\!\!\!/ - m + i\epsilon}$$
$$\Leftrightarrow \qquad \widetilde{D}_F(x-y) = \int d^4\bar{k}\ \frac{i}{k\!\!\!/ - m + i\epsilon}\ e^{-ik\cdot(x-y)}.$$

In the last step, we inverted the Fourier transformation.

15.5.2 Generating Functional for the Free Dirac Field

This calculation will be the analogue to the one for the real scalar field from (>15.2.4). We will leave the $i\epsilon$'s aside here, but technically, they are there.

We plug in the Lagrangian $\mathcal{L} = \mathcal{L}_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ into the generating functional,

$$Z[\bar{\eta},\eta] \coloneqq \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\exp\left(i\int d^4x\,(\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta)\right),$$

and shift the integration variables by

$$\psi' \coloneqq \psi + \underbrace{\int d^4y\left(-i\widetilde{D}_F(x-y)\right)\,\eta(y)}_{=:I}, \qquad \psi' \coloneqq \psi + \underbrace{\int d^4y\left(-i\widetilde{D}_F(x-y)\right)^*\,\bar{\eta}(y)}_{=:\bar{I}}.$$

Expanding the exponent after performing this shift, we get a whole bunch of terms:

$$
\begin{aligned}
&\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta \\
&\qquad = (\bar{\psi}' - \bar{I})(i\partial\!\!\!/ - m)(\psi' - I) + \bar{\eta}(\psi' - I) + (\bar{\psi}' - \bar{I})\eta \\
&\qquad = \bar{\psi}'(i\partial\!\!\!/ - m)\psi' - \bar{\psi}'(i\partial\!\!\!/ - m)I - \bar{I}(i\partial\!\!\!/ - m)\psi' + \bar{I}(i\partial\!\!\!/ - m)I + \bar{\eta}\psi' - \bar{\eta}I + \bar{\psi}'\eta - \bar{I}\eta.
\end{aligned}
$$

Let's evaluate the second, third and fourth of those terms individually, recalling, that $-i\widetilde{D}_F$ is the Greens function of $(i\partial\!\!\!/ - m)$:

$$
\bar{\psi}'(i\partial\!\!\!/ - m)I = \bar{\psi}'(i\partial\!\!\!/ - m)\int d^4y\left(-i\widetilde{D}_F(x-y)\right)\eta(y) = \bar{\psi}'\eta,
$$

$$
\begin{aligned}
\bar{I}(i\partial\!\!\!/ - m)\psi' &= \int d^4y\left(-i\widetilde{D}_F(x-y)\right)^*\bar{\eta}(y)(i\partial\!\!\!/ - m)\psi' \\
&= \int d^4y(-i\partial\!\!\!/ - m)\left(-i\widetilde{D}_F(x-y)\right)^*\bar{\eta}(y)\psi' = \int d^4y\left((i\partial\!\!\!/ - m)\left(-i\widetilde{D}_F(x-y)\right)\right)^*\bar{\eta}(y)\psi' \\
&= \bar{\eta}\psi',
\end{aligned}
$$

$$
\bar{I}(i\partial\!\!\!/ - m)I = \bar{I}\,(i\partial\!\!\!/ - m)\int d^4y\left(-i\widetilde{D}_F(x-y)\right)\eta(y) = \bar{I}\eta.
$$

For the second of those three terms, we made use of partial integration. Plugging those results back in, we find

$$
\begin{aligned}
&\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta \\
&\qquad = \bar{\psi}'(i\partial\!\!\!/ - m)\psi' - \bar{\psi}'\eta - \bar{\eta}\psi' + \bar{I}\eta + \bar{\eta}\psi' - \bar{\eta}I + \bar{\psi}'\eta - \bar{I}\eta \\
&\qquad = \bar{\psi}'(i\partial\!\!\!/ - m)\psi' - \bar{\eta}I.
\end{aligned}
$$

Since we only performed a simple shift, we can replace $\mathcal{D}\bar{\psi}\,\mathcal{D}\psi \to \mathcal{D}\bar{\psi}'\,\mathcal{D}\psi'$ and we find

$$
\begin{aligned}
Z[\bar{\eta},\eta] &:= \int \mathcal{D}\bar{\psi}'\,\mathcal{D}\psi'\exp\left(i\int d^4x\,(\bar{\psi}'(i\partial\!\!\!/ - m)\psi' - \bar{\eta}I)\right) = Z[0,0]\exp\left(-i\int d^4x\,\bar{\eta}I\right) \\
&= Z[0,0]\exp\left(-i\int d^4x\,d^4y\,\bar{\eta}(x)\left(-i\widetilde{D}_F(x-y)\right)\eta(y)\right) \\
&= Z[0,0]\exp\left(-\int d^4x\,d^4y\,\bar{\eta}(x)\,\widetilde{D}_F(x-y)\,\eta(y)\right).
\end{aligned}
$$

15.5.3 Evaluating 2-Point Function with the Generating Functional

Using the explicit formula for $Z[\bar{\eta},\eta]$ for the free Dirac field theory, we find

$$
\begin{aligned}
&\frac{1}{Z[0,0]}\left(-i\frac{\delta}{\delta\bar{\eta}(x_1)}\right)\left(i\frac{\delta}{\delta\eta(x_2)}\right)Z[\eta,\bar{\eta}]\Big|_{\bar{\eta},\eta=0} \\
&\qquad = \frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\exp\left(-\int d^4x\,d^4y\,\bar{\eta}(x)\,\widetilde{D}_F(x-y)\,\eta(y)\right)\Big|_{\bar{\eta},\eta=0} \\
&\qquad = \frac{\delta}{\delta\bar{\eta}(x_1)}\frac{\delta}{\delta\eta(x_2)}\left(-\int d^4x\,d^4y\,\bar{\eta}(x)\,\widetilde{D}_F(x-y)\,\eta(y)\right)\Big|_{\bar{\eta},\eta=0} \\
&\qquad = \frac{\delta}{\delta\bar{\eta}(x_1)}\left(\int d^4x\,d^4y\,\bar{\eta}(x)\,\widetilde{D}_F(x-y)\,\delta(y-x_2)\right)\Big|_{\bar{\eta},\eta=0} \\
&\qquad = \int d^4x\,\delta(x-x_1)\,\widetilde{D}_F(x-x_2)\Big|_{\bar{\eta},\eta=0} = \widetilde{D}_F(x_1-x_2).
\end{aligned}
$$

We used the fact, that we were able to expand the exponential of Graßmann numbers exactly to first order. When evaluating the derivative with respect to $\eta(x_2)$, we had use the differentiation rule

$$
\frac{\delta}{\delta\eta(x_2)}\bar{\eta}(x)\eta(y) = -\bar{\eta}(x)\frac{\delta}{\delta\eta(x_2)}\eta(y) = -\bar{\eta}(x)\,\delta(x_2-y),
$$

which brought us an extra minus sign.

15.7 The Schwinger-Dyson Equations

15.7.1 Taylor Expansion of a Functional

We want to proof the Taylor expansion

$$\mathcal{L}[\phi'(x)] = \mathcal{L}[\phi(x) + \epsilon(x)] = \mathcal{L}[\phi(x)] + \epsilon(x)\frac{\delta}{\delta\phi(x)}\int d^4x'\ \mathcal{L}[\phi(x')] + \mathcal{O}(\epsilon^2)$$

"by example". We use the abbreviation $\phi' \coloneqq \phi + \epsilon$. That is, we want to check whether it is true for the free scalar field Lagrangian $\mathcal{L} = \left(\partial_\mu\phi\right)(\partial^\mu\phi)/2 - m^2\phi^2/2$. Our first task will be to compute the expansion of this particular Lagrangian. We use partial integration here, which is okay, since we will only need this expansion when the Lagrangian is placed in an d^4x-integral:

$$\begin{aligned}\mathcal{L}[\phi'] &= \frac{1}{2}\left((\partial_\mu\phi')(\partial^\mu\phi') - m^2\phi'^2\right) = -\frac{1}{2}\phi'(\Box + m^2)\phi' = -\frac{1}{2}(\phi+\epsilon)(\Box + m^2)(\phi+\epsilon)\\ &= -\frac{1}{2}(\phi(\Box + m^2)\phi + \epsilon(\Box + m^2)\phi + \phi(\Box + m^2)\epsilon) = \mathcal{L}[\phi] - \epsilon(\Box + m^2)\phi.\end{aligned}$$

Alright, so let's see if our above given formula yields the same result. For that purpose, we first calculate

$$\begin{aligned}\frac{\delta}{\delta\phi(x)}\int d^4x'\ \mathcal{L}[\phi(x')] &= -\frac{1}{2}\frac{\delta}{\delta\phi(x)}\int d^4x'\ \phi(x')(\Box_{x'} + m^2)\phi(x')\\ &= -\frac{1}{2}\left(\int d^4x'\ \delta(x'-x)(\Box_{x'} + m^2)\phi(x') + \int d^4x'\ \phi(x')\frac{\delta}{\delta\phi(x)}(\Box_{x'} + m^2)\phi(x')\right)\\ &= -\frac{1}{2}\left((\Box_x + m^2)\phi(x) + \int d^4x'\ (\Box_{x'} + m^2)\phi(x')\frac{\delta}{\delta\phi(x)}\phi(x')\right)\\ &= -\frac{1}{2}\left((\Box_x + m^2)\phi(x) + (\Box_x + m^2)\phi(x)\right) = -(\Box + m^2)\phi(x).\end{aligned}$$

Thus, the right-hand side of our equation reads

$$\mathcal{L}[\phi(x)] + \epsilon(x)\frac{\delta}{\delta\phi(x)}\int d^4x'\ \mathcal{L}[\phi(x')] = \mathcal{L}[\phi] - \epsilon(\Box + m^2)\phi,$$

which is exactly would we also found with the direct calculation.

15.7.2 Derivation of the Schwinger-Dyson Equations

We are going to derive the Schwinger-Dyson equation for the 3-point function of the real scalar field; they will take on the same form also for any other theory and any n-point function.

The 3-point function of the real scalar field is known from section 14.2; it reads

$$\langle\Omega|\mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)|\Omega\rangle = \frac{1}{Z[0]}\int \mathcal{D}\phi\ \exp\left(i\int d^4x\ \mathcal{L}[\phi]\right)\ \phi(x_1)\phi(x_2)\phi(x_3).$$

We can perform a shift $\phi(x) \to \phi(x) + \epsilon(x) \eqqcolon \phi'(x)$, which leaves the measure $\mathcal{D}\phi$ unchanged. The step of shifting the integral is the following equal sign:

$$\int \mathcal{D}\phi\ \exp\left(i\int d^4x\ \mathcal{L}[\phi]\right)\ \phi(x_1)\phi(x_2)\phi(x_3) = \int \mathcal{D}\phi\ \exp\left(i\int d^4x\ \mathcal{L}[\phi']\right)\ \phi'(x_1)\phi'(x_2)\phi'(x_3).$$

We want to consider only terms up to order ϵ of this equation. On the left-hand side, we have only an ϵ^0-order term. On the right-hand side, we find

$$\begin{aligned}\phi'(x_1)\phi'(x_2)\phi'(x_3) &= \big(\phi(x_1)+\epsilon(x_1)\big)\big(\phi(x_2)+\epsilon(x_2)\big)\big(\phi(x_2)+\epsilon(x_2)\big)\\ &= \phi(x_1)\phi(x_2)\phi(x_3)+\epsilon(x_1)\phi(x_2)\phi(x_3)+\phi(x_1)\epsilon(x_2)\phi(x_3)+\phi(x_1)\phi(x_2)\epsilon(x_3)+\mathcal{O}(\epsilon^2)\end{aligned}$$

and

$$\begin{aligned}\exp\left(i\int d^4x\,\mathcal{L}[\phi']\right) &= \exp\left(i\int d^4x\,\mathcal{L}[\phi+\epsilon]\right)\\ &= \exp\left(i\int d^4x\left(\mathcal{L}[\phi(x)]+\epsilon(x)\frac{\delta}{\delta\phi(x)}\int d^4x'\,\mathcal{L}\,[\phi(x')]+\mathcal{O}(\epsilon^2)\right)\right)\\ &= \underbrace{\exp\left(i\int d^4x\,\mathcal{L}[\phi(x)]\right)}_{=e^{iS[\phi(x)]}}\left(1+i\int d^4x\,\epsilon(x)\frac{\delta}{\delta\phi(x)}\int d^4x'\,\mathcal{L}\,[\phi(x')]+\mathcal{O}(\epsilon^2)\right).\end{aligned}$$

Here, at the second equal sign, we used the functional analogue of a Taylor expansion of $\mathcal{L}[\phi+\epsilon]$ for small ϵ. Putting those two results of the right-hand side together, the order ϵ^0 is equal to the left-hand side and what remains is only the order ϵ:

$$\begin{aligned}0 = \int \mathcal{D}\phi\, e^{iS[\phi(x)]}\Bigg(i\int d^4x\,\epsilon(x)\left(\frac{\delta}{\delta\phi(x)}\int d^4x'\,\mathcal{L}\,[\phi(x')]\right)\phi(x_1)\phi(x_2)\phi(x_3)\\ +\epsilon(x_1)\phi(x_2)\phi(x_3)+\phi(x_1)\epsilon(x_2)\phi(x_3)+\phi(x_1)\phi(x_2)\epsilon(x_3)\Bigg).\end{aligned}$$

Substituting $\epsilon(x_i)=\int d^4x\,\epsilon(x)\,\delta(x-x_i)$, we can put also the three terms in the end underneath the d^4x-integral. After that, we can get rid of the $\epsilon(x)$ and d^4x-integral, since the right-hand side must vanish for any $\epsilon(x)$. Finally, we put the first of the four terms in the large bracket on the other side of the equation and multiply by i:

$$\begin{aligned}&\int \mathcal{D}\phi\, e^{iS[\phi(x)]}\left(\frac{\delta}{\delta\phi(x)}\int d^4x'\,\mathcal{L}\,[\phi(x')]\right)\phi(x_1)\phi(x_2)\phi(x_3)\\ &\quad= \int \mathcal{D}\phi\, e^{iS[\phi(x)]}\big(i\delta(x-x_1)\phi(x_2)\phi(x_3)+\phi(x_1)i\delta(x-x_2)\phi(x_3)\\ &\quad+\phi(x_1)\phi(x_2)i\delta(x-x_3)\big).\end{aligned}$$

To see what happens next, let's for one step insert the example for the free Klein-Gordon field derived in the footnote earlier in the current section, that is we plug in $-(\Box+m^2)\phi(x)$ in the first of the first terms:

$$\begin{aligned}&\int \mathcal{D}\phi\, e^{iS[\phi(x)]}\big(-(\Box+m^2)\phi(x)\big)\phi(x_1)\phi(x_2)\phi(x_3)\\ &\quad= \int \mathcal{D}\phi\, e^{iS[\phi(x)]}\big(i\delta(x-x_1)\phi(x_2)\phi(x_3)+\phi(x_1)i\delta(x-x_2)\phi(x_3)\\ &\quad+\phi(x_1)\phi(x_2)i\delta(x-x_3)\big).\end{aligned}$$

We can now pull the operator in front of the functional integral $\Box+m^2$ (of course, it still only acts on the single field ϕ with variable x, not the $\phi(x)$ in the Lagrangian in the exponential (we maybe should've called this integration variable x differently). Anything else is nothing but n-point functions (recall that $S=\int d^4x\,\mathcal{L}$) and we find

$$\begin{aligned}&-(\Box+m^2)\langle\Omega|\mathcal{T}\,\phi(x)\phi(x_1)\phi(x_2)\phi(x_3)|\Omega\rangle\\ &\quad=\langle\Omega|\mathcal{T}\,i\delta(x-x_1)\,\phi(x_2)\phi(x_3)|\Omega\rangle+\langle\Omega|\mathcal{T}\,\phi(x_1)\,i\delta(x-x_2)\,\phi(x_3)|\Omega\rangle\\ &\quad+\langle\Omega|\mathcal{T}\,\phi(x_1)\phi(x_2)\,i\delta(x-x_3)|\Omega\rangle.\end{aligned}$$

Note, that we were only able to pull the term $\Box+m^2$ out of the integral and thus out of the n-point function, after we wrote it in this explicit way; we were not able to pull out its general pendant, the functional derivative over the integral of the Lagrangian. Still, we want to write this last equation also in the general case, but we can only do so by defining a tedious notation. We write the formula as

$$\left\langle \left(\frac{\delta}{\delta\phi(x)} \int d^4x' \,\mathcal{L}\,[\phi(x')] \right) \phi(x_1)\phi(x_2)\phi(x_3) \right\rangle$$
$$= \langle i\delta(x-x_1)\,\phi(x_2)\phi(x_3)\rangle + \langle \phi(x_1)\, i\delta(x-x_2)\,\phi(x_3)\rangle + \langle \phi(x_1)\phi(x_2)\, i\delta(x-x_3)\rangle.$$

To be in consistence with the example we just calculated, the angular brackets denote a time-ordered correlation function in which derivatives on $\varphi(x)$ are placed outside the time-ordering symbol (and, if one wishes, also outside the whole correlation function).

When we generalize this to an arbitrary number of fields of arbitrary types (we call them $\varphi = \phi, A, \psi, \ldots$), we find what are called the *Schwinger-Dyson equations:*

$$\left\langle \left(\frac{\delta}{\delta\varphi(x)} \int d^4x' \,\mathcal{L}\,[\varphi(x')] \right) \varphi(x_1)\cdots\varphi(x_3) \right\rangle = \sum_{i=1}^{n} \langle \varphi(x_1)\cdots i\delta(x-x_i)\cdots\varphi(x_n)\rangle.$$

15.7.3 Noether's Current Conservation

To derive the Schwinger-Dyson equation we used the invariance of the functional integrals under a simple shift $\varphi \to \varphi' = \varphi + \epsilon$ of the fields. That is, ϵ can be an arbitrary function, but still this procedure is limited, since ϵ cannot contain ϕ. For example, $\varphi \to \varphi' = e^{i\alpha}\varphi = \varphi + i\alpha\varphi$, that is $\epsilon = i\alpha\varphi$, is *not* a simple shift.

Still, the action can be invariant under such transformations as well and it is therefore worth to take a look on how to handle them. Let's consider general transformations

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu, \qquad \varphi_a(x) \to \varphi'_a(x') = \varphi_a(x) + \delta\varphi_a(x)$$

for coordinates x^μ and a set of fields φ_a. We already did this once, back then in section 3.2, when we derived the general case of Noether's theorem. What we found was that the action transforms under this transformation like

$$S \to S + \delta S, \qquad \delta S = \int \delta(d^4x\,\mathcal{L}) = \int d^4x\,\partial_\mu \delta j^\mu, \qquad \delta j^\mu = -\mathcal{T}^\mu{}_\nu\,\delta x^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\,\delta\varphi_a,$$

and the current is conserved: $\partial_\mu \delta j^\mu = 0$.

If we write our transformations in dependence of an infinitesimal parameter $\delta\omega$, that is $\delta x^\mu = \delta\omega\,\Delta x^\mu$ and $\delta\varphi_a(x) = \delta\omega\,\Delta\varphi_a(x)$, then also

$$j^\mu = \frac{\delta j^\mu}{\delta\omega} = -\mathcal{T}^\mu{}_\nu\,\Delta x^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\,\Delta\varphi_a.$$

If we now take $\omega \equiv \omega(x)$ to be x-dependent, this also changes the current such that $\partial_\mu j^\mu \neq 0$ and also the action is not conserved anymore:[1] $\delta S = \int d^4x\, j^\mu(x)\partial_\mu\delta\omega(x)$.

The action is nor more invariant. Let's assume that the transformation of the fields $\varphi'_a(x') = \varphi_a(x) + \delta\omega(x)\,\Delta\varphi_a(x)$ is a unitary transformation. Unitary transformations have the property that they do not change the integration measure, thus $\mathcal{D}\varphi'_a \to \mathcal{D}\varphi_a$. We than can go back to the following equation right at the beginning of (>15.7.2) as it still holds for our general unitary transformation:

$$\int \mathcal{D}\varphi\; e^{iS[\varphi(x)]}\,\varphi_a(x_1)\varphi_b(x_2)\varphi_c(x_3) = \int \mathcal{D}\varphi\; e^{iS[\varphi'(x')]}\,\varphi'_a(x'_1)\varphi'_b(x'_2)\varphi'_c(x'_3).$$

---

[1] Let's check this for the example of the Dirac Lagrangian $\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi$. We know from section 3.4 that it is invariant under the transformation $\delta\psi = -i\,\delta\omega\,\psi$, yielding the current $j^\mu = \bar\psi\gamma^\mu\psi$. Let's see what happens if we choose $\delta\omega \equiv \delta\omega(x)$ to be x-dependent:

$$\mathcal{L}[\psi'] = (\bar\psi + i\delta\omega\bar\psi)(i\partial\!\!\!/ - m)(\psi - i\delta\omega\psi) = \mathcal{L}[\psi] - i\bar\psi(i\partial\!\!\!/)\delta\omega\psi + i\delta\omega\bar\psi(i\partial\!\!\!/)\psi$$
$$= \mathcal{L}[\psi] + \bar\psi\big((\partial\!\!\!/\delta\omega)\psi + \delta\omega\partial\!\!\!/\psi\big) - \delta\omega\bar\psi\partial\!\!\!/\psi = \mathcal{L}[\psi] + \bar\psi\gamma^\mu\psi\partial_\mu\delta\omega = \mathcal{L}[\psi] + j^\mu\partial_\mu\delta\omega.$$

Plugging in $S[\varphi'(x')] = S[\varphi(x)] + \delta S[\varphi(x)]$ with the δS given above, we find

$$\begin{aligned}
&\int \mathcal{D}\varphi\, e^{iS[\varphi(x)]}\, \varphi_a(x_1)\varphi_b(x_2)\varphi_c(x_3) \\
&\qquad = \int \mathcal{D}\varphi\, e^{iS[\varphi(x)]} \exp\left(i\int d^4x\, j^\mu(x)\, \partial_\mu \delta\omega(x)\right)\, \varphi'_a(x'_1)\varphi'_b(x'_2)\varphi'_c(x'_3).
\end{aligned}$$

Then we expand the equation on both sides up to the first order in $\delta\omega$. As in (>15.7.2), the order $\delta\omega^0$ on the left- and right-hand side are equal and what remains is only a zero on the left-hand side and the order $\delta\omega^1$ on the right hand side. There is a $\delta\omega$ in the term with the current and in each of the three dashed fields. The first order in $\delta\omega$ on the right-hand side therefore contains four terms each of which gets its $\delta\omega$ from either the current or one of the three fields. As in (>15.7.2) we put the first term (with the $\delta\omega$ coming from the current) on the left-hand side:

$$\begin{aligned}
&\int \mathcal{D}\varphi\, e^{iS[\varphi(x)]} \left(-i\int d^4x\, j^\mu(x)\, \partial_\mu\delta\omega(x)\right)\varphi_a(x_1)\varphi_b(x_2)\varphi_c(x_3) \\
&\qquad = \int \mathcal{D}\varphi\, e^{iS[\varphi(x)]} \big(\delta\omega(x_1)\, \Delta\varphi_a(x_1)\, \varphi_b(x_2)\varphi_c(x_3) + \varphi_a(x_1)\, \delta\omega(x_2)\, \Delta\varphi_b(x_2)\, \varphi_c(x_3) \\
&\qquad\qquad + \varphi_a(x_1)\varphi_b(x_2)\, \delta\omega(x_3)\, \Delta\varphi_c(x_3)\big).
\end{aligned}$$

Now we perform partial integration in the d^4x-integral to get from $j^\mu\partial_\mu\delta\omega$ to $-\delta\omega\partial_\mu j^\mu$. Then, we plug in $\delta\omega(x_i)\, \Delta\varphi_a(x_i) = \int d^4x\, \delta\omega(x)\, \Delta\varphi(x)\, \delta(x - x_i)$ in the three terms on the right hand side. Since $\delta\omega(x)$ is arbitrary, we can get rid of it together with the d^4x-integral on both sides of the equation:

$$\begin{aligned}
&\int \mathcal{D}\varphi\, e^{iS[\varphi(x)]} \left(i\partial_\mu j^\mu(x)\right)\varphi_a(x_1)\varphi_b(x_2)\varphi_c(x_3) \\
&\qquad = \int \mathcal{D}\varphi\, e^{iS[\varphi(x)]}\big(\Delta\varphi_a(x)\delta(x - x_1)\, \varphi_b(x_2)\varphi_c(x_3) + \varphi_a(x_1)\, \Delta\varphi_b(x)\delta(x - x_2)\, \varphi_c(x_3) \\
&\qquad\qquad + \varphi_a(x_1)\varphi_b(x_2)\, \Delta\varphi_c(x)\delta(x - x_3)\big).
\end{aligned}$$

Next, we multiply the equation with $-i$. Also, by our definition of the meaning of the angular brackets in this context, we can write this equation as

$$\begin{aligned}
&\langle \partial_\mu j^\mu(x)\, \varphi_a(x_1)\varphi_b(x_2)\varphi_c(x_3)\rangle \\
&\qquad = \langle \Delta\varphi_a(x)\, (-i)\delta(x - x_1)\, \varphi_b(x_2)\varphi_c(x_3)\rangle + \langle \varphi_a(x_1)\, \Delta\varphi_b(x)\, (-i)\delta(x - x_2)\, \varphi_c(x_3)\rangle \\
&\qquad\qquad + \langle \varphi_a(x_1)\varphi_b(x_2)\, \Delta\varphi_c(x)\, (-i)\delta(x - x_1)\rangle.
\end{aligned}$$

For an arbitrary number of fields, we find correspondingly

$$\langle \partial_\mu j^\mu(x)\, \varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)\rangle = \sum_{i=1}^{n}\langle \varphi_{a_1}(x_1)\cdots\, \Delta\varphi_{a_i}(x)\, (-i)\delta(x - x_i)\cdots\, \varphi_{a_n}(x_n)\rangle.$$

15.7.4 Ward-Takahashi Identity

In QED, the transformation $\psi \to e^{-i\alpha}\psi = (1 - i\alpha)\psi = \psi - i\alpha\psi$ with $\Delta\psi = -i\psi, \Delta\bar{\psi} = i\bar{\psi}$ yields the current

$$j^\mu = \bar{\psi}\gamma^\mu\psi,$$

as we have seen in section 3.4. Thus, the Schwinger-Dyson equations for the 2-point function read (we can set $x \to x_i$, if there is a corresponding δ-function nearby)

$$\begin{aligned}
&\langle \partial_\mu j^\mu(x)\, \psi(x_1)\bar{\psi}(x_2)\rangle = \langle \Delta\psi(x)\, (-i)\delta(x - x_1)\, \bar{\psi}(x_2)\rangle + \langle \psi(x_1)\, \Delta\bar{\psi}(x)\, (-i)\delta(x - x_2)\rangle \\
&\qquad = -i\langle \psi(x_1)\, (-i)\delta(x - x_1)\, \bar{\psi}(x_2)\rangle + i\langle \psi(x_1)\, \bar{\psi}(x_2)\, (-i)\delta(x - x_2)\rangle \\
&\qquad = -\big(\delta(x - x_1) - \delta(x - x_2)\big)\langle \psi(x_1)\bar{\psi}(x_2)\rangle
\end{aligned}$$

$$\Leftrightarrow \quad \langle \partial_\mu j^\mu(x)\, \psi(x_1)\bar{\psi}(x_2)\rangle = -\big(\delta(x - x_1) - \delta(x - x_2)\big)\langle \psi(x_1)\bar{\psi}(x_2)\rangle$$

$$\Leftrightarrow \quad \partial_\mu \langle 0 | \mathcal{T}\, j^\mu(x)\, \psi(x_1)\bar{\psi}(x_2) | 0 \rangle = -\big(\delta(x - x_1) - \delta(x - x_2)\big) \langle 0 | \mathcal{T}\, \psi(x_1)\bar{\psi}(x_2) | 0 \rangle.$$

Multiplying by $e^{-ik\cdot x}\, e^{iq\cdot x_1}\, e^{-ip\cdot x_2}$ and integrating over x, x_1, x_2 yields on the left-hand side (we perform partial integration with the ∂_μ)

$$\begin{aligned}
&\int d^4x\, d^4x_1\, d^4x_2\, e^{-ik\cdot x}\, e^{iq\cdot x_1}\, e^{-ip\cdot x_2}\, \partial_\mu \langle 0 | \mathcal{T}\, j^\mu(x)\, \psi(x_1)\bar{\psi}(x_2) | 0 \rangle \\
&\quad = \underbrace{-(-ik_\mu)}_{=ik_\mu} \int d^4x\, d^4x_1\, d^4x_2\, e^{-ik\cdot x}\, e^{iq\cdot x_1}\, e^{-ip\cdot x_2}\, \langle 0 | \mathcal{T}\, j^\mu(x)\, \psi(x_1)\bar{\psi}(x_2) | 0 \rangle \\
&\quad = g^{-1} k_\mu \int d^4x\, d^4x_1\, d^4x_2\, e^{-ik\cdot x}\, e^{iq\cdot x_1}\, e^{-ip\cdot x_2}\, \langle 0 | \mathcal{T}\, igj^\mu(x)\, \psi(x_1)\bar{\psi}(x_2) | 0 \rangle = g^{-1} k_\mu \mathcal{M}^\mu.
\end{aligned}$$

Note, that $igj^\mu = ig\bar{\psi}\gamma^\mu\psi$ contains the vertex factor. Thus, this is exactly the amplitude $\mathcal{M}^\mu$ for two external electrons with one vertex, but without an external photon field – that is, there is no polarization vector ε^μ of the external photon, but instead there is a k_μ in front. This is exactly what we had for the Ward-Takahashi identity.

Now, we consider the right-hand side: Also here, we perform the integration over x, x_1, x_2 after multiplying by the exponential functions:

$$\begin{aligned}
&-\int d^4x\, d^4x_1\, d^4x_2\, e^{-ik\cdot x}\, e^{iq\cdot x_1}\, e^{-ip\cdot x_2}\, \big(\delta(x - x_1) - \delta(x - x_2)\big) \langle 0 | \mathcal{T}\, \psi(x_1)\bar{\psi}(x_2) | 0 \rangle \\
&\quad = -\int d^4x_1\, d^4x_2\, e^{-i(k-q)\cdot x_1}\, e^{-ip\cdot x_2}\, \langle 0 | \mathcal{T}\, \psi(x_1)\bar{\psi}(x_2) | 0 \rangle \\
&\qquad\qquad - \int d^4x_1\, d^4x_2\, e^{-i(k+p)\cdot x_2}\, e^{iq\cdot x_1}\, \langle 0 | \mathcal{T}\, \psi(x_1)\bar{\psi}(x_2) | 0 \rangle \\
&\quad = -\mathcal{M}_0(p; q - k) + \mathcal{M}_0(p + k; q).
\end{aligned}$$

Again, we get amplitudes (to see this, also recall the alternative LSZ reduction formulas from section 7.5. Hence, the equation reads

$$k_\mu \mathcal{M}^\mu(k; p; q) = -g\big(\mathcal{M}_0(p, q - k) - \mathcal{M}_0(p + k; q)\big).$$

16 SYSTEMATIC RENORMALIZATION

16.1 Superficial Degree of Divergence

16.1.1 Formula for the Superficial Degree of Divergence (QED)

We already found that the superficial degree of divergence can be given as

$$D = d \cdot L - P_\mathrm{e} - 2P_\gamma.$$

Recall, that in our original Feynman rules given in section, each internal momentum (that is each propagators) comes with a momentum integral and each vertex with a δ-function, eliminating one of the integrals. The remaining integrals are loop integrals and their number is the number of loops. Thus, the number of loops L (remining integrals) is the number of propagators (initial integrals) minus the number of vertices (δ-functions). Well, actually, there is one δ-function left in the end, the one that ensured total momentum conservation. As it is not use to eliminate integrals, should take $V-1$ instead of V:

$$L = P_\mathrm{e} + P_\gamma - (V-1).$$

Also, each external photon is connected to one vertex, each internal photon is connected to two vertices. And each vertex is only connected to exactly one photon:

$$V = 2P_\gamma + N_\gamma.$$

By the same argument, we have $2V = 2P_\mathrm{e} + N_\mathrm{e}$.

Let's now plug in the L from above into the formula for D and then we plug in $P_\gamma = V/2 - N_\gamma/2$ and $P_\mathrm{e} = V - N_\mathrm{e}/2$:

$$\begin{aligned}
D &= d \cdot \Big(P_\mathrm{e} + P_\gamma - (V-1)\Big) - P_\mathrm{e} - 2P_\gamma \\
&= d \cdot \left(\left(V - \frac{1}{2}N_\mathrm{e}\right) + \left(\frac{1}{2}V - \frac{1}{2}N_\gamma\right) - (V-1)\right) - \left(V - \frac{1}{2}N_\mathrm{e}\right) - 2\left(\frac{1}{2}V - \frac{1}{2}N_\gamma\right) \\
&= \frac{1}{2}d \cdot (-N_\mathrm{e} + V - N_\gamma + 2) - 2V + \frac{1}{2}N_\mathrm{e} + N_\gamma \\
&= d + \left(\frac{d}{2} - 2\right)V + \left(-\frac{d}{2} + 1\right)N_\gamma + \left(-\frac{d}{2} + \frac{1}{2}\right)N_\mathrm{e} = d + \frac{d-4}{2}V - \frac{d-2}{2}N_\gamma - \frac{d-1}{2}N_\mathrm{e}.
\end{aligned}$$

16.1.2 Mass Dimension of the QED Coupling Constant

The action often appears in the exponent of an exponential function like $\exp(iS)$. Thereby, it must be dimensionless in natural units. In those units, d^dx has the dimension m^{-d} (see section 1.2), which we will simply call "mass dimension $-d$" for now and write $[dx] = -1$ and $[d^dx] = d[dx] = -d$. Since $S = \int d^dx\, \mathcal{L}$, the Lagrangian must have mass dimension d, which we will write as $[\mathcal{L}] = d$. Consider the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial - m)\psi + g\bar{\psi}\gamma^\mu\psi A_\mu, \qquad F^{\mu\nu}F_{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big).$$

Obviously, we have from the kinetic energy of the photon

$$d = [\mathcal{L}] = \big[F^{\mu\nu}F_{\mu\nu}\big] = [(\partial^\mu A^\nu)^2] = 2[\partial^\mu] + 2[A^\nu] = 2 + 2[A^\nu] \qquad \Leftrightarrow \qquad [A^\nu] = \frac{d-2}{2},$$

since $[\partial^\mu] = -[dx] = 1$. The fermionic term yields, assuming $[\psi] = [\bar{\psi}]$,

$$d = [\mathcal{L}] = [\bar{\psi}m\psi] = [m] + 2[\psi] = 1 + 2[\psi] \qquad \Leftrightarrow \qquad [\psi] = \frac{d-1}{2}.$$

Knowing the dimensions of the fields, we can finally use the interaction term to find the dimension of the coupling constant:

$$d = [\mathcal{L}] = \left[g\bar{\psi}\psi A_\mu\right] = [g] + 2[\psi] + \left[A_\mu\right] = [g] + 2\frac{d-1}{2} + \frac{d-2}{2} \qquad \Leftrightarrow \qquad [g] = -\frac{d-4}{2}.$$

16.1.3 Formula for the Superficial Degree of Divergence (ϕ^4 Theory)

Let N be the number of external lines, P the number of propagators and V the number of vertices. Let L be the number of Loops. Since each vertex has n lines attached to it, we have

$$nV = N + 2P,$$

since in total we have nV attached to some vertex. A propagator is attached to two vertices, an external line to one. By the same argument as in (>16.1.1), the number of loops can be given as

$$L = P - (V - 1).$$

The propagator in ϕ^4 theory has two powers of momentum in the denominator (as we know from section 8.1). Each loop contributes d powers of momentum in the numerator. Thus, the superficial degree of divergence reads

$$D = dL - 2P = d\left(\frac{1}{2}(nV - N) - (V-1)\right) - (nV - N) = d + \left(n\frac{d-2}{2} - d\right)V - \frac{d-2}{2}N.$$

In the same way is in (>16.1.2), we want to determine the mass dimension of the coupling constant λ. The kinetic term yields the mass dimension of the field:

$$d = [\mathcal{L}] = \left[(\partial_\mu\phi)^2\right] = 2 + 2[\phi] \qquad \Leftrightarrow \qquad [\phi] = \frac{d-2}{2}.$$

Thus,

$$d = [\mathcal{L}] = [\lambda\phi^n] = [\lambda] + n\frac{d-2}{2} \qquad \Leftrightarrow \qquad [\lambda] = -\left(n\frac{d-2}{2} - d\right).$$

16.2 Divergent QED Amplitudes

16.2.1 Furry's Theorem

In QED, S-matrix elements can be written as a Fourier transformed n-point function of the external Heisenberg fields by LSZ reduction (see section 7.5) and those n-point functions can in turn be written as correlation functions of the external interaction picture fields together with an exponential function of the interaction Lagrangian (see section 7.9). Thus, in S-matrix elements, we will encounter correlation functions of the form

$$\langle 0|\mathcal{T} \quad \text{(external fields)} \quad \exp(i\textstyle\int d^4x\, \mathcal{L}_{\text{Int}})\,|0\rangle,$$

where $\mathcal{L}_{\text{Int}} = g\bar{\psi}\gamma^\mu\psi A_\mu$ (with $g = e > 0$). In the n-th order of perturbation theory (that is the n-th order of the Taylor expansion of the exponential function), we have the following contribution from the exponential:

$$\frac{1}{n!}\left(i\int d^4x\, \mathcal{L}_{\text{Int}}\right)^n = \frac{(ig)^n}{n!}\int d^4x_1 \cdots d^4x_n \;\; \bar{\psi}(x_1)\gamma^{\mu_1}\psi(x_1)A_{\mu_1}(x_1)\cdots\bar{\psi}(x_n)\gamma^{\mu_n}\psi(x_n)A_{\mu_n}(x_n).$$

The photon fields commute with the fermion fields and we can then separate the correlation functions, such that we have one with all the photon fields and one with all the fermion fields. Furry's theorem applies to amplitudes with external photons only, thus we assume that the only fermion fields come from the exponential. The correlation function of the Fermion fields then reads

$$\langle 0|\bar{\psi}(x_1)\gamma^{\mu_1}\psi(x_1)\cdots\bar{\psi}(x_n)\gamma^{\mu_n}\psi(x_n)|0\rangle = \langle 0|j^{\mu_1}(x_1)\cdots j^{\mu_n}(x_n)|0\rangle,$$

using the fermion current $j^\mu = \bar{\psi}\gamma^\mu\psi$.

Any *internal* photon is attached to two vertices. Thus, there must be an even number of vertices with only internal photons attached to them. Each external photon is attached to one additional vertex. Thus, the total number of vertices is even or odd for an even or odd number of external photons respectively (in other words: for an odd number of external photons all orders of perturbation theory with even order will vanish).

Hence, for an odd number of external photons, we have only diagrams with a correlation function of an odd number of fermion currents; that is n is odd.

If we use now that the current transforms under charge conjugation by $Cj^\mu C^\dagger = -j^\mu$, we can plug in a pair of $C^\dagger C$ between all the currents, use $C|0\rangle = |0\rangle$ and thus

$$\langle 0|j^{\mu_1}(x_1)\cdots j^{\mu_n}(x_n)|0\rangle = \langle 0|C^\dagger C j^{\mu_1}(x_1)C^\dagger\cdots Cj^{\mu_n}(x_n)C^\dagger C|0\rangle = -\langle 0|C^\dagger j^{\mu_1}(x_1)\cdots j^{\mu_n}(x_n)C|0\rangle.$$

For an odd number of currents, we get a total minus sign. Thus, this correlation function vanishes and thereby all diagrams with an odd number of external photons.

16.2.2 The Electron Self-Energy

Let's call the amputated amplitude (that is, without the external fermion spinors, since they do not appear underneath a potentially diverging loop integral) of the amputated electron self-energy diagram $\mathcal{F}$. According to Feynman rules, this amplitude is a function of the slashed electron momentum $\not{p}$, so let's Taylor expand it:

$$\mathcal{F}(\not{p}) = F_0 + F_1\not{p} + F_2\not{p}^2 + \cdots, \qquad F_n = \frac{1}{n!}\frac{d^n\mathcal{F}(\not{p})}{d\not{p}^n}\bigg|_{\not{p}=0}.$$

After applying the δ-functions of the vertices, the momenta appearing in the denominators of the propagators are a combination of the external momentum p and one or more loop momenta, which we denote here as k. And the amplitude is a sum of products of such propagators. When differentiating with respect to $\not{p}$, we get several terms from the product rule and in each term, there will be a derivative

$$\frac{d}{d\not{p}}\frac{1}{\not{p}+\not{k}-m} = -\frac{1}{(\not{p}+\not{k}-m)^2}.$$

That is, each derivative with respect to $\not{p}$ lowers the superficial degree of divergence (the power of k) by 1. Since $F_0 = \mathcal{F}(\not{p}=0)$, F_0 has the same superficial divergence like $\mathcal{F}$, namely $D = 1$. Thus, F_1 is superficially logarithmic divergent and all $F_{n\geq 2}$ are finite. That is, $F_0 \sim \Lambda$ and $F_1 \sim \ln\Lambda$. However, the actual divergence (in contrast to the superficial one) of F_0 is also just $\sim \ln\Lambda$: We know this, since we explicitly found in section 11.5

$$\Sigma_{\not{p}} = \frac{\alpha}{2\pi}\int_0^1 dx\,(2m - x\not{p})\ln\frac{x\Lambda^2}{\Delta} + \mathcal{O}(\alpha^2)$$

(note that $m = m_0 + \mathcal{O}(\alpha)$).

16.3 Counter Term Renormalization

16.3.1 Counter Terms

Recall from section 13.5, that we can write the full bare propagators and the full bare vertex factor as the finite renormalized propagators and the finite renormalized vertex factor times factors of Z_i:

$$\frac{i}{\not p - m_0 - \Sigma(\not p)} = \frac{iZ_2}{\not p - m - \Sigma_R(\not p)}, \qquad \text{where} \quad \Sigma_R = \Sigma(\not p) - \Delta m - \delta_2(\not p - m) + \mathcal{O}(\alpha^2),$$

$$\frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1-\Pi(q^2)} = \frac{-i\eta^{\mu\nu}}{q^2}\frac{Z_3}{1-\Pi_R(q^2)}, \qquad \text{where} \quad \Pi_R = \Pi(q^2) - \delta_3 + \mathcal{O}(\alpha^2),$$

$$ig_0\Gamma^\mu(q) = \frac{ig}{Z_1\sqrt{Z_3}}\Gamma_R^\mu(q), \qquad \text{where} \quad \Gamma_R^\mu(q) = \gamma^\mu\left(1 + \delta F_1(q^2) + \delta_1^{(2)} + \mathcal{O}(\alpha^2)\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

In chapter 13, we defined the δ_i as the corrections to the field strength renormalization: $Z_i = 1 + \delta_i$. A quantity with superscript (2) like $\delta_i^{(2)}$ denotes its order-α contribution. Let us also define δ_m as follows:

$$\delta_m := Z_2 m_0 - m \qquad \Leftrightarrow \qquad m_0 = \frac{1}{Z_2}(m + \delta_m) = m + \delta_m^{(2)} - m\delta_2^{(2)} + \mathcal{O}(\alpha^2)$$

$$\Longrightarrow \qquad \Delta m := m - m_0 = -\delta_m^{(2)} + m\delta_2^{(2)} + \mathcal{O}(\alpha^2)$$

$$\Longrightarrow \qquad \Sigma_R(\not p) = \Sigma^{(2)}(\not p) + \delta_m - \delta_2 m - \delta_2(\not p - m) + \mathcal{O}(\alpha^2) = \Sigma^{(2)}(\not p) + \delta_m - \delta_2\not p + \mathcal{O}(\alpha^2).$$

Let us expand the renormalized electron propagator (this is just the derivation of (>13.3.2) backwards) in α:

$$\begin{aligned}\frac{i}{\not p - m_0 - \Sigma(\not p)} &= \frac{iZ_2}{\not p - m - \Sigma_R(\not p)}\\ &= Z_2\left(\frac{i}{\not p - m} + \frac{i}{\not p - m}\left(-i\Sigma^{(2)}(\not p)\right)\frac{i}{\not p - m} + \frac{i}{\not p - m}i\left(\delta_2^{(2)}\not p - \delta_m^{(2)}\right)\frac{i}{\not p - m} + \mathcal{O}(\alpha^2)\right).\end{aligned}$$

The same thing for the photon propagator reads (backwards computation of (>13.4.7))

$$\begin{aligned}\frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1-\Pi(q^2)} &= \frac{-i\eta^{\mu\nu}}{q^2}\frac{Z_3}{1-\Pi_R(q^2)}\\ &= Z_3\left(\left(\frac{-i\eta_{\mu\nu}}{q^2}\right) + \left(\frac{-i\eta_{\mu\rho}}{q^2}\; i\Pi_R^{(2)\rho\sigma}(q)\; \frac{-i\eta_{\sigma\nu}}{q^2}\right) + \mathcal{O}(\alpha^2)\right)\\ &= Z_3\left(\left(\frac{-i\eta_{\mu\nu}}{q^2}\right) + \left(\frac{-i\eta_{\mu\rho}}{q^2}\; i(q^2\eta^{\rho\sigma} - q^\rho q^\sigma)\Pi_R^{(2)}(q)\; \frac{-i\eta_{\sigma\nu}}{q^2}\right) + \mathcal{O}(\alpha^2)\right)\\ &= Z_3\left(\left(\frac{-i\eta_{\mu\nu}}{q^2}\right) + \left(\frac{-i\eta_{\mu\rho}}{q^2}\; i\Pi^{(2)\rho\sigma}(q)\; \frac{-i\eta_{\sigma\nu}}{q^2}\right)\right.\\ &\qquad\left. + \left(\frac{-i\eta_{\mu\rho}}{q^2}\; i(q^2\eta^{\rho\sigma} - q^\rho q^\sigma)\left(-\delta_3^{(2)}\right)\; \frac{-i\eta_{\sigma\nu}}{q^2}\right) + \mathcal{O}(\alpha^2)\right).\end{aligned}$$

Similarly, the expansion of the vertex correction reads

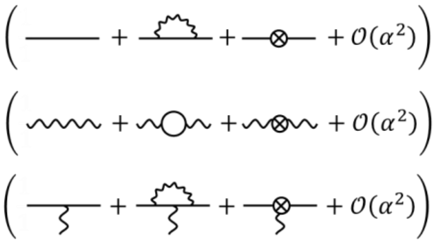
$$\begin{aligned}ig_0\Gamma^\mu(q) &= \frac{ig}{Z_1\sqrt{Z_3}}\Gamma_R^\mu(q) = \frac{ig}{Z_1\sqrt{Z_3}}\left(\gamma^\mu F_1^{(2)}(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2^{(2)}(q^2) + \gamma^\mu\delta_1^{(2)} + \mathcal{O}(\alpha^2)\right)\\ &= \frac{1}{Z_1\sqrt{Z_3}}\left(\Gamma^{(2)\mu}(q) + ig\gamma^\mu\delta_1^{(2)} + \mathcal{O}(\alpha^2)\right).\end{aligned}$$

Recall that $-i\Sigma^{(2)}$ is just an electron line with a single photon loop, $i\Pi^{(2)\rho\sigma}$ is the fermion loop of the photon self-energy and $\Gamma^{(2)\mu}$ is the first correction to the QED vertex. Recall that we can use the physical masses and charges instead of the bare masses and charges in these terms, since the difference is of order α^2. Let us therefore give the three equations above pictorially as follows:

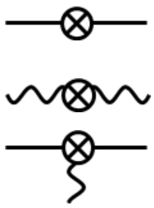
$$\frac{i}{\not{p} - m_0 - \Sigma(\not{p})} = \frac{iZ_2}{\not{p} - m - \Sigma_R(\not{p})} = Z_2 \cdot \left(\cdots + \mathcal{O}(\alpha^2) \right)$$

$$\frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1-\Pi(q^2)} = \frac{-i\eta^{\mu\nu}}{q^2}\frac{Z_3}{1-\Pi_R(q^2)} = Z_3 \cdot \left(\cdots + \mathcal{O}(\alpha^2) \right)$$

$$ig_0\Gamma^\mu = \frac{ig}{Z_1\sqrt{Z_3}}\Gamma_R^\mu = \frac{1}{Z_1\sqrt{Z_3}} \cdot \left(\cdots + \mathcal{O}(\alpha^2) \right)$$



Here, the third term in each bracket is called a *counter term*; comparing with the equations above, we define such Feynman diagram lines as



$$:= i\left(\delta_2^{(2)}\not{p} - \delta_m^{(2)}\right),$$

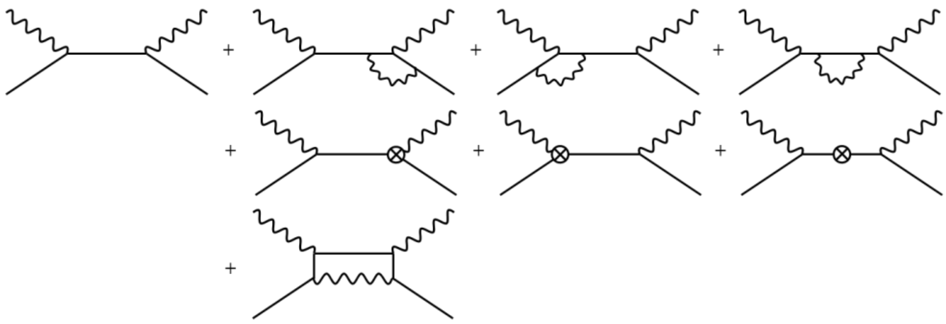
$$:= -i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta_3^{(2)},$$

$$:= ig\gamma^\mu\delta_1^{(2)}.$$

Note, that the $\delta_i^{(2)}$ are of order α; hence they do never appear at leading order computations.

We have seen in (>13.5.7) that the factors Z_i of the propagator will precisely cancel the factors Z_i of the vertex factor. That is, we can effectively "ignore" these factors and simply compute the Feynman diagrams in the brackets as propagators and as the vertex factor to NLO. The counter terms will cancel the divergences of to one loop diagrams.

For example, Compton scattering will contain the following diagrams to order α^2 (compare this diagram expansion to the one in (>13.5.7)):



16.3.2 The Renormalized Lagrangian with Counter Terms

The original QED Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_0^{\mu\nu}F_{0\mu\nu} + \bar{\psi}_0(i\not{\partial} - m_0)\psi_0 + g_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu}.$$

Here, we did not only equip the mass and the charge with an index 0, but also the fields. That is, this Lagrangian depends on the bare mass, the bare charge and the bare fields.

Let us then introduce *renormalized fields* ψ, A^μ without an index 0 as

$$\psi_0 = \sqrt{Z_2}\psi, \qquad A_0^\mu = \sqrt{Z_3}A^\mu.$$

In terms of the renormalized field, the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + Z_2 \bar{\psi}(i\partial\!\!\!/ - m_0)\psi + Z_2\sqrt{Z_3} g_0 \bar{\psi}\gamma^\mu \psi A_\mu$$
$$= -\frac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(Z_2 i\partial\!\!\!/ - Z_2 Z_m m)\psi + Z_1 g \bar{\psi}\gamma^\mu \psi A_\mu,$$

where we have also plugged in

$$g \coloneqq \frac{Z_2\sqrt{Z_3}}{Z_1} g_0, \qquad m_0 = Z_m m.$$

Let us now use the following relations between the Z_i's and the δ_i's:

$$Z_1 = 1 + \delta_1, \qquad Z_2 = 1 + \delta_2, \qquad Z_3 = 1 + \delta_3,$$
$$Z_2 Z_m m = m + \delta_m.$$

Plugging in these relations, we find

$$\mathcal{L} = -\frac{1}{4}(1 + \delta_3) F^{\mu\nu} F_{\mu\nu} + \bar{\psi}\big((1 + \delta_2) i\partial\!\!\!/ - (m + \delta_m)\big)\psi + (1 + \delta_1) g \bar{\psi}\gamma^\mu \psi A_\mu$$
$$= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi + g\bar{\psi}\gamma^\mu \psi A_\mu$$
$$-\frac{1}{4}\delta_3 F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(\delta_2 i\partial\!\!\!/ - \delta_m)\psi + \delta_1 g \bar{\psi}\gamma^\mu \psi A_\mu.$$

The first three terms are precisely the terms of the standard QED Lagrangian, with the fields replaced by the renormalized fields and the physical mass and charge instead of the bare mass and bare charge. That is quite nice, however there is a downside: We have also three new terms in the Lagrangian, called "counter terms".

16.3.3 Mass Term as a Perturbation

It may seem odd to take the first two terms of the counter Lagrangian as a perturbation; after all, they are no interactions but a kinetic term and a mass term. In fact, it does not make a difference, if a term in the Lagrangian is taken as a perturbation or not, as long it is computed to the sufficient order of perturbation theory.

To get a better understanding of this fact, consider a free scalar theory

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 - \frac{m^2}{2}\phi^2.$$

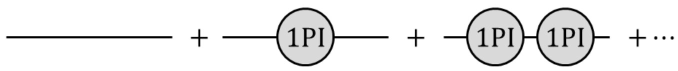
In section 8.1 we found that the propagator of this theory reads

$$\frac{i}{q^2 - m^2}.$$

Let's now see what we get if we treat the mass term perturbatively as a interaction. In this case, the propagator would only come from the kinetic term $(\partial^\mu \phi)^2/2$. This kinetic term can also be thought of a the Lagrangian of a free *massless* scalar field. Therefore, the propagator then is just i/q^2.

In 8.1 we found that an interaction term $-\lambda\phi^4/4!$ yields a vertex with four attached propagators; it comes with the Feynman rule $-i\lambda$. In analogy, our mass term $-m^2\phi^2/2!$ can be treated as a vertex with two attached propagators and a vertex factor $-im^2$.

Since this is the only interaction of our Lagrangian, the full propagator is as usual (like in chapter 12)

full propagator =


with the one-particle irreducible simply containing one of the "mass vertices". That is,

$$\text{full propagator} = \frac{i}{q^2} + \frac{i}{q^2}(-im^2)\frac{i}{q^2} + \frac{i}{q^2}(-im^2)\frac{i}{q^2}(-im^2)\frac{i}{q^2} + \cdots = \frac{i}{q^2}\sum_{n=0}^{\infty}\left((-im^2)\frac{i}{q^2}\right)^n$$
$$= \frac{i}{q^2}\frac{1}{1-(-im^2)\frac{i}{q^2}} = \frac{i}{q^2-m^2},$$

which is exactly the same propagator, as if we had taken the mass term as a mass term in the first place. Thus, it is absolutely consistent, to treat terms like mass terms (but also kinetic terms) as perturbations. If they don't contain the coupling constant (like in the example above): We have to expand them to infinity or can equivalently handle them exact (as we did always before). If they do contain the coupling constant (like for the mass counter term): A expansion up to the necessary order in the coupling constant is sufficient.

That is, as long we expand the perturbation term to sufficient order in the coupling constant, we can treat any term as a perturbation. Thus, it is definitely always valid to treat the counter terms as perturbations, since they contain coupling constants.

16.3.4 Counter Electron Propagator from Renormalized Lagrangian

We know that a Lagrangian term $\bar{\psi}_0(i\not\partial - m_0)\psi_0$ yields a fermion propagator with the Feynman rule $i/(\not p - m_0)$ exactly. Thus, we expect that a Lagrangian with the two terms

$$\bar{\psi}(i\not\partial - m)\psi + \bar{\psi}(\delta_2 i\not\partial - \delta_m)\psi = \bar{\psi}\big((1+\delta_2)i\not\partial - (m+\delta_m)\big)\psi$$

yields a propagator with Feynman rule

$$\frac{i}{(1+\delta_2)\not p - (m+\delta_m)}$$

to all orders of perturbation theory. Expanding it to NLO yields

$$\frac{i}{(1+\delta_2)\not p - (m+\delta_m)} = \frac{i}{\not p - m + (\delta_2\not p - \delta_m)}$$
$$= \frac{i}{\not p - m}\frac{i}{1 - i(\delta_2\not p - \delta_m)\frac{i}{\not p - m}}$$
$$= \frac{i}{\not p - m}\left(1 + i\left(\delta_2^{(2)}\not p - \delta_m^{(2)}\right)\frac{i}{\not p - m}\right) + \mathcal{O}(\alpha^2)$$
$$= \frac{i}{\not p - m} + \frac{i}{\not p - m}i\left(\delta_2^{(2)}\not p - \delta_m^{(2)}\right)\frac{i}{\not p - m} + \mathcal{O}(\alpha^2).$$

Thus, the counter term of our newly constructed Lagrangian produces nothing but a vertex, connecting two fermion lines, with the vertex factor $i(\delta_2\not p - \delta_m)$. Thus, our newly constructed Lagrangian does indeed yield the Feynman rule

 $:= i\left(\delta_2^{(2)}\not p - \delta_m^{(2)}\right).$

16.3.5 Counter Photon Propagator from Renormalized Lagrangian

We know that a Lagrangian term $-F_0^{\mu\nu}F_{0\mu\nu}$ yields a photon propagator with the Feynman rule $-i\eta^{\mu\nu}/q^2$ exactly. Thus, we expect that a Lagrangian with the two terms

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}\delta_3 F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}(1+\delta_3)F^{\mu\nu}F_{\mu\nu}$$

yields a propagator with Feynman rule

$$\frac{-i\eta^{\mu\nu}}{q^2(1+\delta_3)},$$

very similar to (>16.3.4).[1]

In (>13.4.7) we found that

$$\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ i\Pi^{\rho\sigma}(q)\ \ \frac{-i\eta_{\sigma\nu}}{q^2}\right)+\cdots=\frac{-i}{q^2}\left(\eta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{1-\Pi(q^2)}+\frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2},$$

where terms $\sim q^\mu q^\nu$ effectively vanish due to the Ward identity (this is also explained in (>13.4.7)). Note also, that $\Pi^{\rho\sigma}(q)=(\eta^{\rho\sigma}q^2-q^\rho q^\sigma)\Pi(q^2)$. Thus, effectively, this equation can be written as

$$\frac{-i\eta_{\mu\nu}}{q^2}\frac{1}{1-\Pi(q^2)}=\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ i(\eta^{\rho\sigma}q^2-q^\rho q^\sigma)\Pi^{(2)}(q^2)\ \ \frac{-i\eta_{\sigma\nu}}{q^2}\right)+\mathcal{O}(\alpha^2).$$

Hence,

$$\frac{-i\eta_{\mu\nu}}{q^2(1+\delta_3)}=\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)+\left(\frac{-i\eta_{\mu\rho}}{q^2}\ \ i(\eta^{\rho\sigma}q^2-q^\rho q^\sigma)\left(-\delta_3^{(2)}\right)\ \ \frac{-i\eta_{\sigma\nu}}{q^2}\right)+\mathcal{O}(\alpha^2)$$

and thus, the Feynman rule

 $:= -i(q^2\eta^{\mu\nu}-q^\mu q^\nu)\delta_3^{(2)},$

indeed follows from our newly constructed renormalized Lagrangian.

16.3.6 Counter Vertex Factor from Renormalized Lagrangian

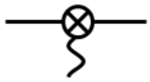
We know that a Lagrangian term $g_0\bar{\psi}_0\gamma^\mu\psi_0A_{0\mu}$ yields a photon propagator with the Feynman rule $ig_0\gamma^\mu$. Thus, the term

$$g\delta_1\bar{\psi}\gamma^\mu\psi A_\mu$$

obviously yields a term

$$ig\delta_1\gamma^\mu$$

and thus, the Feynman rule

 $:= ig\gamma^\mu\delta_1^{(2)}.$

indeed follows from our newly constructed renormalized Lagrangian.

16.4 Renormalization Conditions

16.4.1 On-Shell Renormalization Conditions

In section 13.5 with found that the full interacting fermion propagator can be given as

$$\frac{i}{\not{p}-m_0-\Sigma(\not{p})}=\frac{iZ_2}{\not{p}-m-\Sigma_R(\not{p})},$$

where the factor Z_2 can be cancelled with a factor Z_2^{-1} that appears in the vertex corrections (>13.5.7).

In section 13.5 we have seen explicitly that $\Sigma_R(\not{p})$ obeys $\Sigma_R(\not{p}=m)=0$. This ensures that the pole of the renormalized propagator indeed lies at the physical mass m. Let us now turn around the argument

[1] If you are not convinced that this is right, take the kinetic counter term of the photon and perform partial integration:

$$-\frac{1}{4}\delta_3F^{\mu\nu}F_{\mu\nu}\overset{\text{PI}}{=}\frac{1}{2}A_\mu(-\delta_3\Box\eta^{\mu\nu}-\delta_3\partial^\mu\partial^\nu)A_\nu.$$

Then, δ_3 appears as a proportionality factor of $\Box$ and the propagator – that is the Green's function of this operator – will have this factor δ_3 in front of its q^2.

and impose $\Sigma_R(\not{p} = m) = 0$ as a *condition*. Using the explicit form of $\Sigma_R(\not{p})$ from (>16.3.1), this condition implies

$$\Sigma_R(m) = \left(\Sigma^{(2)}(\not{p}) + \delta_m^{(2)} - \delta_2^{(2)}\not{p}\right)\Big|_{\not{p}=m} + \mathcal{O}(\alpha^2) = \Sigma^{(2)}(m) + \delta_m^{(2)} - \delta_2^{(2)}m + \mathcal{O}(\alpha^2) \stackrel{!}{=} 0.$$

Analogous to (>13.3.4), the condition $\partial\Sigma_R(\not{p})/\partial\not{p}|_{\not{p}=m} = 0$ ensures that the residue of the pole of the propagator is iZ_2 (after cancellation of Z_2 only i). This condition implies

$$\frac{\partial\Sigma_R(\not{p})}{\partial\not{p}}\bigg|_{\not{p}=m} = \frac{\partial}{\partial\not{p}}\left(\Sigma^{(2)}(\not{p}) + \delta_m^{(2)} - \delta_2^{(2)}\not{p}\right)\bigg|_{\not{p}=m} + \mathcal{O}(\alpha^2) = \frac{\partial\Sigma^{(2)}(\not{p})}{\partial\not{p}}\bigg|_{\not{p}=m} - \delta_2^{(2)} + \mathcal{O}(\alpha^2) \stackrel{!}{=} 0.$$

We can now use these two conditions to *fix* the counter terms as

$$\delta_2^{(2)} = \frac{\partial\Sigma^{(2)}(\not{p})}{\partial\not{p}}\bigg|_{\not{p}=m}, \qquad \delta_m^{(2)} = \delta_2^{(2)}m - \Sigma^{(2)}(m).$$

Of course, $\Sigma^{(2)}$ can still be known only by explicit computation of loop corrections. Note that – by construction – this fixing yields the same result for $\delta_2^{(2)}$ as in section 13.3. Thus, fixing the δ's by the use of the renormalization conditions

$$\Sigma_R(p = m) = 0, \qquad \frac{\partial\Sigma_R(\not{p})}{\partial\not{p}}\bigg|_{\not{p}=m} = 0$$

is a perfectly consistent way to find explicit expressions for the counter term δ's.

Similarly, the pole of the full photon propagator

$$\frac{-i\eta_{\mu\nu}}{q^2}\frac{Z_3}{1 - \Pi_R(q^2)},$$

also known from section 13.5 (where the factor Z_3 is again cancelled against a vertex correction), has a pole at $q^2 = 0$ due to the fact that $\Pi_R(q^2 = 0) = 0$ (see section 13.5). Again, we can *impose* $\Pi_R(q^2 = 0) = 0$ as a *renormalization condition* which fixes $\delta_3^{(2)}$:

$$\Pi_R(q^2 = 0) = \Pi^{(2)}(q^2 = 0) - \delta_3^{(2)} \stackrel{!}{=} 0.$$

Finally, we have found in section 13.2 that $\delta_1^{(2)} = -\delta F_1(q^2 = 0)$. This fixing of $\delta_1^{(2)}$ is achieved by the renormalization condition

$$\Gamma_R^\mu(q = 0) = \gamma^\mu,$$

since

$$\Gamma_R^\mu(q = 0) = \gamma^\mu\left(1 + \delta F_1(0) + \delta_1^{(2)}\right) + 0 \stackrel{!}{=} \gamma^\mu \qquad \Leftrightarrow \qquad \delta_1^{(2)} = -\delta F_1(0).$$

16.4.2 Explicit Formulas for the Counter Terms

In dimensional renormalization, we find (without proof)

$$\begin{aligned}
\delta_1 &= \delta_2 \\
&= -\frac{\mu^{4-d} e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2-d/2)}{((1-x)^2 m^2 + x\nu^2)^{2-d/2}} \\
&\qquad \left((2-\epsilon)x - \frac{\epsilon}{2} \frac{2x(1-x)m^2}{(1-x)^2 m^2 + x\nu^2} \big(4 - 2x - \epsilon(1-x)\big) \right) \\
&= -\frac{e^2}{8\pi^2} \left(\frac{1}{\epsilon} + \frac{1}{2} \ln \frac{\tilde{\mu}^2}{m^2} - 2 - \ln \frac{\nu^2}{m^2} + \mathcal{O}(\epsilon) \right),
\end{aligned}$$

where m is the electron and ν an artificial photon mass. Note that the terms with an ϵ in the first formula for $\delta_1 = \delta_2$ do not vanish for $\epsilon \to 0$, since the Γ-function contains a $1/\epsilon$ pole (see section 13.2). Also,

$$\delta_3 = -\frac{\mu^{4-d} e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2-d/2)}{(m^2)^{2-d/2}} \cdot 8x(1-x) = -\frac{4}{3} \frac{e^2}{(4\pi)^2} \left(\frac{2}{\epsilon} + \ln \frac{\tilde{\mu}^2}{m^2} + \mathcal{O}(\epsilon) \right).$$

16.5 About the Charge Renormalization

16.5.1 Equality of the first two Renormalization Factors

From section 11.6 we know $\delta Z_2 = -\delta F_1(0)$. Thus,

$$\begin{aligned}
&\delta Z_2 = -\delta F_1(0) \\
\Leftrightarrow \quad & 1 + \delta Z_2 = 1 - \delta F_1(0) \\
\Leftrightarrow \quad & Z_2 = 1 + \delta_1 \\
\Leftrightarrow \quad & Z_2 - 1 = \delta_1 \\
\Leftrightarrow \quad & \delta_2 = \delta_1.
\end{aligned}$$

We used here the definition $\delta_2 \coloneqq Z_2 - 1$ from section 15.3 and the observation $\delta_1 = -\delta F_1(0)$ from section 15.4.

17 The Renormalization Group

17.1 Analogy to Statistical Mechanics

17.1.1 Wick Rotation in the Generating Functional

We know from section 14.2 that the generating functional of scalar fields reads

$$Z[J] \coloneqq \int \mathcal{D}\phi \exp\left(i \int d^4x \left(\mathcal{L} + J(x)\phi(x)\right)\right),$$

where – in ϕ^4-theory – the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4.$$

In section 12.4 we introduced the Wick rotation applied to four-momentum integrals; however, there is no reason why it should not also be applicable to Minkowski space coordinates. We substitute $x^0 = -ix_E^0$ and $\vec{x} = \vec{x}_E$, such that $x^2 = (x^0)^2 - \vec{x}^2 = -(x_E^0)^2 - |\vec{x}_E|^2 = -x_E^2$. Since

$$(\partial^\mu \phi)^2 = (\partial^0 \phi)^2 - (\nabla \phi)^2 = \left(\frac{1}{-i}\partial_E^0 \phi\right)^2 - (\nabla_E \phi)^2 = -(\partial_E^0 \phi)^2 - (\nabla_E \phi)^2 = -\left(\partial_E^\mu \phi\right)^2$$

and $d^4x = -id^4x_E$, we find

$$\mathcal{L} = -\frac{1}{2}\left(\partial_E^\mu \phi\right)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 =: -\mathcal{L}_E$$

and thus

$$\begin{aligned} Z[J] &\coloneqq \int \mathcal{D}\phi \exp\left(-ii \int d^4x_E \left(-\mathcal{L}_E + J(x_E)\phi(x_E)\right)\right) \\ &= \int \mathcal{D}\phi \exp\left(-\int d^4x_E \left(\mathcal{L}_E - J(x_E)\phi(x_E)\right)\right). \end{aligned}$$

17.1.2 Euclidean Correlation Function

We will now perform the analogous steps to (>15.2.4) for our Euclidean description. Using partial integration, the Euclidean Lagrangian of the free Klein-Gordon field can be written as

$$\mathcal{L}_E = \frac{1}{2}\left(\partial_E^\mu \phi\right)^2 + \frac{m^2}{2}\phi^2 = -\frac{1}{2}\phi(\Box_E - m^2)\phi.$$

In (>15.2.4) we shifted the fields by (some integral of) iD_F, which was helpful because of the fact that iD_F is the Greens function of the Klein-Gordon operator $\Box + m^2$. Hence, in the present case we use D_E instead of iD_F, where

$$D_E(x_E) \coloneqq -\int d^4\bar{k}_E \frac{e^{ik_E \cdot x_E}}{k_E^2 + m^2}$$

is the Greens function of $\Box_E - m^2$:

$$(\Box_E - m^2)D_E(x_E) = -\int d^4\bar{k}_E \frac{-k_E^2 - m^2}{k_E^2 + m^2} e^{ik_E \cdot x_E} = \delta(x_E).$$

Similarly to (>15.2.4), we shift our field (which is the integration parameter of the functional integral) like

$$\phi' := \phi + \underbrace{\int d^4y_E\, D_E(x_E - y_E)\, J(y_E)}_{=:I_E}.$$

Note that we have a plus sign here in contrast to (>15.2.4), since also the factor $J\phi$ in the generating functional from (>17.1.1) has the opposite sign as back then. This shift yields (using the abbreviation $G_E := \Box_E - m^2$)

$$\begin{aligned}\mathcal{L}_E - J\phi &= -\frac{1}{2}(\phi' - I_E)G_E(\phi' - I_E) - J(\phi' - I_E)\\ &= -\frac{1}{2}(\phi' G_E\phi' - \phi' G_E I_E - I_E G_E \phi' + I_E G_E I_E) - J\phi' + JI_E.\end{aligned}$$

Since D_E appearing in I_E is the Greens function of G_E, just as iD_F appearing in I was the Greens function of G in (>15.2.4) by exactly the same derivations as back then we find

$$\phi' G_E I_E = \phi' J, \qquad I_E G_E \phi' = \phi' J, \qquad I_E G_E I_E = JI_E.$$

Thus,

$$\mathcal{L}_E - J\phi = -\frac{1}{2}(\phi' G_E \phi' - 2\phi' J + JI_E) - J\phi' + JI_E = -\frac{1}{2}\phi' G_E \phi' + \frac{1}{2}JI_E = \mathcal{L}_E[\phi'] + \frac{1}{2}JI_E.$$

Thus, by this shift, the generating functional of (>17.1.1) can be brought into the form

$$\begin{aligned}Z[J] &= \int \mathcal{D}\phi \exp\Big(-\int d^4x_E\, \big(\mathcal{L}_E - J(x_E)\phi(x_E)\big)\Big) = \int \mathcal{D}\phi' \exp\Big(-\int d^4x_E\, \Big(\mathcal{L}_E[\phi'] + \frac{1}{2}JI_E\Big)\Big)\\ &= Z[0]\exp\Big(-\frac{1}{2}\int d^4x_E \int d^4y_E\, J(x_E)\, D_E(x_E - y_E)\, J(y_E)\Big).\end{aligned}$$

Here, the $\mathcal{L}_E$ part was absorbed into $Z[0]$ and the JI_E part was explicitly written. Thus, by the formula from section 14.2, we find

$$\begin{aligned}\langle\Omega|\mathcal{T}\phi(x_{E1})\phi(x_{E2})|\Omega\rangle &= \frac{1}{Z[0]}\Big(-i\frac{\delta}{\delta J(x_{E1})}\Big)\Big(-i\frac{\delta}{\delta J(x_{E2})}\Big)Z[J]\Big|_{J=0} = D_E(x_{E1} - x_{E2})\\ &= -\int d^4\bar{k}_E \frac{e^{ik_E\cdot(x_{E1}-x_{E2})}}{k_E^2 + m^2}.\end{aligned}$$

The derivation of the last equal sign is exactly the same as in (>15.2.5). In three dimensions this integral can relatively easy be evaluated:

$$\begin{aligned}\int d^3\bar{k}\, \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2 + m^2} &= \frac{2\pi}{(2\pi)^3}\int_0^\infty dk\, k^2 \int_{-1}^1 d\cos\theta\, \frac{e^{ixk\cos\theta}}{k^2 + m^2} = \frac{1}{(2\pi)^2}\int_0^\infty dk\, k\frac{e^{ixk} - e^{-ixk}}{ix}\frac{1}{k^2 + m^2}\\ &= \frac{1}{(2\pi)^2 ix}\Big(\int_0^\infty dk\, \frac{k\, e^{ixk}}{k^2 + m^2} - \int_0^{-\infty} dk\, \frac{k\, e^{ixk}}{k^2 + m^2}\Big) = \frac{1}{(2\pi)^2 ix}\int_{-\infty}^\infty dk\, \frac{k\, e^{ixk}}{k^2 + m^2},\end{aligned}$$

where we have turned $k \to -k$ in the second term of the large bracket. Closing the contour above in the complex plane, there is one pole $k = +im$ enclosed. Using residues theorem from the footnote on page 26, we find

$$\int d^3\bar{k}\, \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2 + m^2} = \frac{1}{(2\pi)^2 ix}\cdot 2\pi i\,(k - im)\frac{k\, e^{ixk}}{(k + im)(k - im)}\Big|_{k=im} = \frac{e^{-mx}}{4\pi x}.$$

In our four-dimensional case the angular integral of spherical coordinates would be a bit more complicated, the property e^{-xm} would, however, not change:

$$\langle\Omega|\mathcal{T}\phi(x_{E1})\phi(x_{E2})|\Omega\rangle \sim e^{-m|x_{E1}-x_{E2}|}.$$

17.2 Wilson's Approach – Effective Lagrangian

17.2.1 Functional Integral over Fourier Components of Fields

Technically, the translation from $\mathcal{D}\phi(x)$ to $\mathcal{D}\phi(k)$ works as follows. Our definition of functional integration from (>15.1.3) was constructed for a one-dimensional function $x(t)$; we discretized t into N steps t_i with spacing ϵ. Then, we integrated over any possible value $x(t_i)$ at one of the fixed times t_i. Finally, we considered the limit $N \to \infty, \epsilon \to 0$.

The role of the functions $x(t)$ now play the fields $\phi(x)$. Therefore, we discretize x and use many discrete points x_i instead, which form a lattice in four-dimensional space. Let their spacing be $2\pi/L$, where L^d is the space-time volume. We now can use *discrete* Four transformation, where we have a sum instead of an integral:

$$\phi(x_i) = \frac{1}{L^d} \sum_n \phi(k_n)\, e^{-ik_n \cdot x_i}.$$

We assume ϕ to be real, such that $\phi(-k_n) = \phi^*(k_n)$. Thus, what we actually mean by a functional integral is

$$\mathcal{D}\phi(x) := \prod_i d\phi(x_i) = \prod_i d\phi(k_n) =: \mathcal{D}\phi(k).$$

The spacing being $2\pi/L$, in reciprocal space each discrete momentum point k_n occupies the space $\Delta^d k := (2\pi/L)^d$. Thus,

$$\phi(x_i) = \frac{1}{L^d} \sum_n \phi(k_n)\, e^{-ik_n \cdot x_i} \frac{\Delta^d k}{(2\pi/L)^d} = \sum_n \phi(k_n)\, e^{-ik_n \cdot x_i} \frac{\Delta^d k}{(2\pi)^d} \to \int d^d\bar{k}\, \phi(k)\, e^{-ik\cdot x} = \phi(x),$$

where the last step represents the limit $L \to \infty$ (that is, the spacing goes to zero: $2\pi/L \to 0$).

Thus, the individual terms in the Lagrangian can be given in terms of the Fourier components as follows:

$$\begin{aligned}
\frac{1}{2}\int d^d x\, (\partial^\mu \phi)^2 &= \frac{1}{2}\int d^d x \left(\int d^d\bar{k}\, (-ik^\mu)\, \phi(k)\, e^{-ik\cdot x} \right)^2 \\
&= -\frac{1}{2}\int d^d x \int d^d\bar{k}\, d^d\bar{q}\, k^\mu q_\mu\, \phi(k)\phi(q)\, e^{-i(k+q)\cdot x} \\
&= -\frac{1}{2}\int d^d\bar{k}\, d^d\bar{q}\, k^\mu q_\mu\, \phi(k)\phi(q)\, (2\pi)^4 \delta(k+q) = \frac{1}{2}\int d^d\bar{k}\, k^2\, |\phi(k)|^2,
\end{aligned}$$

$$\frac{m^2}{2}\int d^d x\, \phi^2 = \frac{1}{2}\int d^d\bar{k}\, |\phi(k)|^2.$$

The term $\sim \phi^4$ can be transformed in the same way. Because it is shorter, we continue to write the exponential in the real space, however, to integrate over $\mathcal{D}\phi(k)$ we keep in mind how the exponent looks like in Fourier space.

17.2.2 Lagrangian in Terms of the two Fields with low and high Momenta

We want to replace ϕ by $\phi + \hat{\phi}$ in the Euclidean Lagrangian (dropping the index E)

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4.$$

Let's do this replacement term by term, starting with the kinetic term:

$$\frac{1}{2}(\partial^\mu \phi)^2 \to \frac{1}{2}\left(\partial^\mu(\phi + \hat{\phi})\right)^2 = \frac{1}{2}(\partial^\mu \phi)^2 + \frac{1}{2}\left(\partial^\mu \hat{\phi}\right)^2.$$

Obviously, there is also a term $(\partial^\mu \phi)(\partial^\mu \hat{\phi})$ when expanding the square. Very similar to the evaluation of $\int d^4x\, (\partial^\mu \phi)^2$ in (>17.2.1), when we Fourier expand the fields, we eventually will get a δ-function $\delta(k+q)$, where k and q are the momenta of the Fourier expansion of ϕ and $\hat{\phi}$ respectively. By definition of ϕ and $\hat{\phi}$ the condition of $k=-q$ can never be met, which is why this term vanishes (underneath a d^4x-integral).

For the same reason, the term $\phi\hat{\phi}$ disappears also in the mass term:

$$\frac{m^2}{2}\phi^2 \to \frac{m^2}{2}\big(\phi+\hat{\phi}\big)^2 = \frac{m^2}{2}\phi + \frac{m^2}{2}\hat{\phi}.$$

Finally, the interaction term yields

$$\begin{aligned}\frac{\lambda}{4!}\phi^4 \to \frac{\lambda}{4!}\big(\phi+\hat{\phi}\big)^4 &= \frac{\lambda}{4!}\big(\phi^4 + 4\phi^3\hat{\phi} + 6\phi^2\hat{\phi}^2 + 4\phi\hat{\phi}^3 + \hat{\phi}^4\big)\\ &= \lambda\left(\frac{1}{4!}\phi^4 + \frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right).\end{aligned}$$

Thus, the Lagrangian becomes

$$\mathcal{L}[\phi] \to \mathcal{L}[\phi] + \frac{1}{2}\big(\partial^\mu\hat{\phi}\big)^2 + \frac{m^2}{2}\hat{\phi} + \lambda\left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right).$$

17.2.3 Propagator of the High-Momentum Field

We will treat also the mass term as a perturbation (if this is surprising, see (>16.3.3)). We have found the propagator of a kinetic term like $(\partial^\mu\hat{\phi})^2$ in the Euclidean Lagrangian already in (>17.1.2) to be[1]

$$D_E(x_E) = \int d^d\bar{k}\,\frac{e^{ik\cdot x}}{k^2}.$$

Since the mass is treated as a perturbation, the propagator is massless. We define the propagator in momentum space by

$$\begin{aligned}\langle\hat{\phi}(k)\hat{\phi}(p)\rangle &:= \int d^dx\, d^dy\, e^{-ik\cdot x}e^{-ip\cdot y}\, D_E(x-y)\\ &= \int d^dx\, d^dy\, e^{-ik\cdot x}e^{-ip\cdot y} \int\limits_{b\Lambda\le|q|\le\Lambda} d^d\bar{q}\,\frac{e^{iq\cdot(x-y)}}{q^2}\\ &= \int\limits_{b\Lambda\le|q|\le\Lambda} \frac{d^d\bar{q}}{q^2}\, d^4x\, d^4y\, e^{-i(k-q)\cdot x}e^{-i(p+q)\cdot y} = \int\limits_{b\Lambda\le|q|\le\Lambda} \frac{d^d\bar{q}}{q^2}\,(2\pi)^d\delta(k-q)\,(2\pi)^d\delta(p+k)\\ &= \frac{(2\pi)^d\delta(k+p)}{k^2}\,\Theta(k),\end{aligned}$$

where $\Theta(k)=1$ for $b\Lambda\le k<\Lambda$ and 0 otherwise.

17.2.4 Effective Mass Correction

When we expand the exponential to first order, there will be a term

$$-\{\text{external fields } \phi\}\frac{\lambda}{4}\int d^dx\, \phi^2\hat{\phi}^2,$$

where the x-integral comes from the Lagrangian. Since only ϕ-fields (in contrast to $\hat{\phi}$-fields) appear as external fields (we assume that the momentum of external fields is always lower than $b\Lambda$), is this

[1] We dropped the indices E here, but the coordinates are still Euclidean. *Well, to be honest, we also dropped an an overall minus sign. Peskin&Schröder does not have this sign anywhere. I will simply drop it here. I hope, it's not too important …*

term only two $\hat{\phi}$-fields appear. Due to Wick's theorem, they will pair and form a propagator; this propagator we have computed in (>17.2.3). Thus, ignoring external fields, we find[1]

$$
\begin{aligned}
-\frac{\lambda}{4}\int d^dx\,\phi^2\hat{\phi}^2 &= -\frac{\lambda}{4}\int d^dx\, d^d\bar{k}\, d^d\bar{p}\,\phi^2\,\langle\hat{\phi}(k)\hat{\phi}(p)\rangle\, e^{ik\cdot x}e^{ip\cdot x} \\
&= -\frac{\lambda}{4}\int d^dx\, d^d\bar{k}\, d^d\bar{p}\,\phi^2(x)\,\frac{(2\pi)^d\delta(k+p)}{k^2}\, e^{ik\cdot x}e^{ip\cdot x} = -\frac{\lambda}{4}\int d^dx\, d^d\bar{k}\,\phi^2(x)\,\frac{1}{k^2} \\
&= -\frac{1}{2}\int d^dx\,\phi^2(x)\underbrace{\left(\frac{\lambda}{2}\int d^4\bar{k}\,\frac{1}{k^2}\right)}_{=:\Delta m^2},
\end{aligned}
$$

where

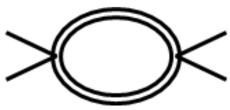
$$
\begin{aligned}
\Delta m^2 &= \frac{\lambda}{2}\int\limits_{b\Lambda\le|k|\le\Lambda}\frac{d^d\bar{k}}{k^2} = \frac{\lambda}{2}\frac{\Omega_d}{(2\pi)^d}\int_{b\Lambda}^{\Lambda}dk\,\frac{k^{d-1}}{k^2} = \frac{\lambda}{2}\frac{\Omega_d}{(2\pi)^d}\left[\frac{k^{d-2}}{d-2}\right]_{b\Lambda}^{\Lambda} = \frac{\lambda}{2}\frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\left[\frac{k^{d-2}}{d-2}\right]_{b\Lambda}^{\Lambda} \\
&= \frac{\lambda}{(4\pi)^{d/2}\Gamma(d/2)}\frac{1-b^{d-2}}{d-2}\Lambda^{d-2}.
\end{aligned}
$$

Here, Ω_d is the surface of a d-dimensional sphere from section 12.6. Note that $\Delta m^2 > 0$ for $d > 2$.

Thus, to the first order in the expansion of the exponential, the $\phi^2\hat{\phi}^2$-term is just a correction to the ϕ-mass term. If we draw $\hat{\phi}$-propagators as a double line, this vertex look as follows:

17.2.5 Effective Interaction Correction

Consider a diagram with two of those vertices:



Obviously, this is a contribution of order λ^2. It comes from a second order expansion of the exponential and contains two term $\phi^2\hat{\phi}^2$. Specifically, this dagram is due to the term

$$
\{\text{external fields } \phi\}\frac{1}{2!}\left(-\frac{\lambda}{4}\int d^dx\,\phi^2\hat{\phi}^2\right)^2.
$$

The factor 1/2! is the factor of the second order Taylor expansion. Again, we neglect the external fields and we find

$$
\frac{1}{2!}\left(-\frac{\lambda}{4}\int d^dx\,\phi^2\hat{\phi}^2\right)^2 = \frac{1}{2!}\frac{\lambda^2}{4^2}\int d^dx\, d^dy\,\phi^2(x)\phi^2(y)\,\hat{\phi}^2(x)\hat{\phi}^2(y).
$$

There are now two relevant contractions: For the diagram above, only $\hat{\phi}$-propagators from x to y (the position of the vertices) contribute. This diagram contains no propagator that returns back to the same vertex (in contrast to the diagram we considered above). Since we have two fields $\hat{\phi}(x)$ and two fields $\hat{\phi}(x)$ there are two possible equivalent contractions. We can just one of them and, since they are

---

[1] Since the integration over momenta are always implied to be in the correct region, that is $b\Lambda \le k < \Lambda$ for $\hat{\phi}$ fields, we can set $\Theta(k) = 1$.
Note also, that we to not replace $\lambda^2 \to \mu^{4-d}\lambda$; thus, λ is not dimensionless in d dimensions (see section 13.2), which is no problem in the present computation. In the whole Wilson's Approach Computation, the explicit factor μ^{4-d} is simply of no use.

equivalent, simply add a factor of 2 for the other. To specify the contractions, we expand the $\hat{\phi}$-fields in Fourier space:

$$
\begin{aligned}
&\frac{1}{2!}\left(-\frac{\lambda}{4}\int d^dx\,\phi^2\hat{\phi}^2\right)^2 \\
&\quad=\frac{2\lambda^2}{2!\,4^2}\int d^dx\,d^dy\,\phi^2(x)\phi^2(y) \\
&\qquad\qquad\left(d^d\bar{k}_1\,d^d\bar{p}_1\,\langle\hat{\phi}(k_1)\hat{\phi}(p_1)\rangle\,e^{ik_1\cdot x}e^{ip_1\cdot y}\right)\left(d^d\bar{k}_2\,d^d\bar{p}_2\,\langle\hat{\phi}(k_2)\hat{\phi}(p_2)\rangle\,e^{ik_2\cdot y}e^{ip_2\cdot x}\right) \\
&\quad=\frac{2\lambda^2}{2!\,4^2}\int d^dx\,d^dy\,\phi^2(x)\phi^2(y) \\
&\qquad\left(d^d\bar{k}_1\,d^d\bar{p}_1\frac{(2\pi)^d\delta(k_1+p_1)}{k_1^2}e^{ik_1\cdot x}e^{ip_1\cdot y}\right)\left(d^d\bar{k}_2\,d^d\bar{p}_2\frac{(2\pi)^d\delta(k_2+p_2)}{k_2^2}e^{ik_2\cdot y}e^{ip_2\cdot x}\right) \\
&\quad=\frac{2\lambda^2}{2!\,4^2}\int d^dx\,d^dy\,\phi^2(x)\phi^2(y)\frac{d^d\bar{k}_1}{k_1^2}\frac{d^d\bar{k}_2}{k_2^2}e^{i(k_1-k_2)\cdot(x-y)}.
\end{aligned}
$$

Since we are only interested in situations, where the external particles have small momenta compared to the virtual $\hat{\phi}$-particles, we can assume that $y\approx x$ (for very large momenta of the virtual particle, the distance between the adjacent vertices becomes very small). Also, we know from momentum conservation that $k_2=-k_1$. When we plug in those relations into the formula above, we are left with two integrals, d^dy and $d^d\bar{k}_2$, to which the integrand is a constant. Since one of those two integrals is spatial, the other reciprocal, they will just cancel each other. Thus, we are left with

$$
\frac{1}{2!}\left(-\frac{\lambda}{4}\int d^dx\,\phi^2\hat{\phi}^2\right)^2=\frac{2\lambda^2}{2!\,4^2}\int d^dx\,\phi^4(x)\frac{d^d\bar{k}_1}{k_1^4}=-\frac{\Delta\lambda}{4!}\int d^dx\,\phi^4(x),
$$

where we introduced the abbreviation

$$
\begin{aligned}
\Delta\lambda &:= \underbrace{-4!\frac{2\lambda^2}{2!\,4^2}}_{=\frac{3\lambda^2}{2}}\int\limits_{b\Lambda\le|k|\le\Lambda}\frac{d^d\bar{k}}{k^4}=-\frac{3\lambda^2}{2}\frac{\Omega_d}{(2\pi)^d}\int_{b\Lambda}^{\Lambda}dk\frac{k^{d-1}}{k^4}=-\frac{3\lambda^2}{2}\frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\left[\frac{1}{d-4}k^{d-4}\right]_{b\Lambda}^{\Lambda} \\
&=-\frac{3\lambda^2}{(4\pi)^{d/2}\Gamma(d/2)}\frac{1-b^{d-4}}{d-4}\Lambda^{d-4}\ \overset{d\to4}{\longrightarrow}\ -\frac{3\lambda^2}{(4\pi)^2}\ln 1/b.
\end{aligned}
$$

To find the final result, we needed to expand the numerator as follows for small $x:=d-4$:

$$
1-b^x=1-e^{x\ln b}\approx 1-(1+x\ln b)=-x\ln b=x\ln 1/b.
$$

17.3 Renormalization Group Flows

17.3.1 Rescaled Effective Lagrangian

Applying the rescaling

$$
k':=k/b,\qquad x'=xb,
$$

to the action

$$
\begin{aligned}
\int d^dx\,\mathcal{L}_{\text{eff}}=\int d^dx\Big(&\frac{1}{2}(1+\Delta Z)\left(\partial_\mu\phi\right)^2+\frac{1}{2}(m^2+\Delta m^2)\phi^2 \\
&+\frac{1}{4!}(\lambda+\Delta\lambda)\phi^4+\Delta C\left(\partial_\mu\phi\right)^4+\Delta D\phi^6+\cdots\Big),
\end{aligned}
$$

we find, using $d^dx=d^dx'\,b^{-d}$ and $\partial_\mu=\partial'_\mu\,b$,

$$
\int d^dx\,\mathcal{L}_{\text{eff}}=\int d^dx'\,b^{-d}\left(\frac{1}{2}(1+\Delta Z)b^2\left(\partial'_\mu\phi\right)^2+\frac{1}{2}(m^2+\Delta m^2)\phi^2\right.
$$

$$+\frac{1}{4!}(\lambda+\Delta\lambda)\phi^4+\Delta C b^4\left(\partial'_\mu\phi\right)^4+\Delta D\phi^6+\cdots\Big).$$

To bring the kinetic term of the effective Lagrangian into the usual form $(\partial_\mu\phi')^2/2$, we also rescale the fields according to

$$\phi' := \sqrt{b^{2-d}(1+\Delta Z)}\phi.$$

Then, we find

$$\int d^dx\,\mathcal{L}_{\text{eff}} = \int d^dx'\left(\frac{1}{2}\left(\partial'_\mu\phi'\right)^2+\frac{m'^2}{2}\phi'^2+\frac{\lambda'}{4!}\phi'^4+\Delta C'\left(\partial'_\mu\phi'\right)^4+\Delta D'\phi'^6+\cdots\right),$$

where

$$m'^2 := (m^2+\Delta m^2)(1+\Delta Z)^{-1}b^{-2},$$

$$\lambda' := (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}b^{d-4},$$

$$\Delta C' := (C+\Delta C)(1+\Delta Z)^{-2}b^{d},$$

$$\Delta D' := (D+\Delta C)(1+\Delta Z)^{-3}b^{2d-6}.$$

Note that, in case of the original Lagrangian that we considered here, $C=D=0$.

17.3.2 General Transformation Behaviour of the Coefficients

The statement is: The coefficient of an operator with n powers of ϕ and m derivatives transforms (in the vicinity of the free-field Lagrangian) like

$$(\text{new coefficient}) = b^\alpha\ (\text{old coefficient}),\quad \text{with}\quad \alpha = d_{nm}-d,\quad d_{nm} := n(d/2-1)+m.$$

We will "proof" this by the examples of the four terms from which we already derived the transformation behaviour:

$$\begin{array}{lllll}
m^2\phi^2 & \Rightarrow\ n=2, m=0 & \Rightarrow\ \alpha=d_{20}-d=-2 & \Rightarrow\ m'^2=m^2b^{-2}, \\
\lambda\phi^4 & \Rightarrow\ n=2, m=0 & \Rightarrow\ \alpha=d_{40}-d=d-4 & \Rightarrow\ \lambda'=\lambda b^{d-4}, \\
C(\partial_\mu\phi)^4 & \Rightarrow\ n=4, m=4 & \Rightarrow\ \alpha=d_{44}-d=d & \Rightarrow\ C'=Cb^d, \\
D\phi^6 & \Rightarrow\ n=6, m=0 & \Rightarrow\ \alpha=d_{60}-d=2d-6 & \Rightarrow\ D'=Db^{2d-6}.
\end{array}$$

The results on the very right side we already got before in section 16.3. Thereby, we assume that the formula is correct.

17.3.3 General Mass Dimension of the Coefficients

From (>16.1.3) we know that in a theory with a kinetic term $(\partial_\mu\phi)^2$ in the Lagrangian, the field has the mass dimension

$$[\phi] = \frac{d-2}{2}.$$

Thus, an operator containing n fields and m derivatives has the mass dimension

$$n\frac{d-2}{2}+m,$$

since $[\partial_\mu]=1$. For example, $\phi^2(\partial_\mu\phi)^2$ has $n=4$ fields and $m=2$ derivatives. Thus, d_{nm} is just the mass dimension of the operator.

Since the Lagrangian has mass dimension $[\mathcal{L}]=d$, the coefficient of the operator has mass dimension

$$d=[\mathcal{L}]=[\text{coefficient}]+d_{nm}\qquad\Leftrightarrow\qquad [\text{coefficient}]=-(d_{nm}-d).$$

17.4 Callan-Symanzik Equation

17.4.1 Derivation of the Callan-Symanzik Equation

Consider the n-point function

$$G^{(n)}(\{x_i\},\lambda,\mu) := \langle\Omega|\mathcal{T}\phi_r(x_1)\cdots\phi_r(x_n)|\Omega\rangle = Z^{-n/2}\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(x_n)|\Omega\rangle,$$

with renormalized fields ϕ_r and bare fields ϕ. We want to investigate a shift in the scale M, which also induces shifts to λ and $\sqrt{Z}$:

$$M\to M+\delta M,\qquad \lambda\to\lambda+\delta\lambda,\qquad \sqrt{Z}\to\sqrt{Z}+\sqrt{Z}\delta\eta.$$

We denoted the shift of $\sqrt{Z}$ as $\sqrt{Z}\delta\eta$, where we wrote a factor of $\sqrt{Z}$ explicitly for convenience. Since the bare n-point function is independent of M (and also λ, Z, since it depends on λ_0 and the bare fields), it should be invariant under this shift and we find

$$\begin{aligned}0 &= \delta\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(x_n)|\Omega\rangle = \delta\big(Z^{n/2}G^{(n)}\big) = G^{(n)}\delta Z^{n/2} + Z^{n/2}\,\delta G^{(n)}\\ &= G^{(n)}nZ^{n/2}\delta\eta + Z^{n/2}\left(\frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G}{\partial\lambda}\delta\lambda\right)\end{aligned}$$

where we used

$$\delta Z^{n/2} = \delta\big(\sqrt{Z}\big)^n = n\big(\sqrt{Z}\big)^{n-1}\underbrace{\delta\sqrt{Z}}_{=\sqrt{Z}\delta\eta} = nZ^{n/2}\delta\eta.$$

We can bring this equation to its usual form by substituting the definitions

$$\beta := M\frac{\delta\lambda}{\delta M},\qquad \gamma := M\frac{\delta\eta}{\delta M};$$

specifically, we plug in $\delta\lambda = \delta M/M\ \beta$ and $\delta\eta = \delta M/M\ \gamma$ and multiply the equation by $M/\delta M$:

$$0 = n\gamma\,G^{(n)} + M\frac{\partial G^{(n)}}{\partial M} + \beta\frac{\partial G^{(n)}}{\partial\lambda} = \left(M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + n\gamma\right)G^{(n)}.$$

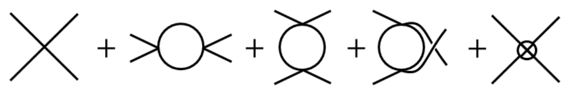
In general, the functions β and γ could depend on λ, M and Λ. However, they are – by their definition – dimensionless and thus cannot depend on M and Λ individually but only on M/Λ. Also, λ, M and Z are independent of the cutoff Λ, thus also β and γ must be independent of Λ. What remains is only the dependence on λ:

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right)G^{(n)}(\{x_i\},\lambda,M) = 0.$$

17.4.2 Renormalized 4-Point Function

The amplitude of a process $p_1p_2\to p_3p_4$ in ϕ^4 theory is given by

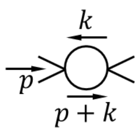
$$i\mathcal{M}(p_1p_2\to p_3p_4) =$$



$$+\,\mathcal{O}(\lambda^3)$$

According to the Feynman rules in section 8.1, the single vertex simply contributes $-i\lambda$ and according to the counter term Feynman rules the counter vertex simply contributes $-i\delta_\lambda$, which will turn out to be of order λ^2.

$$(-i\lambda)^2 iV(s) =$$



Let the amplitude of the s-channel diagram be $(-i\lambda)^2 V(s)$, and we find from the Feynman rules

$$(-i\lambda)^2 iV(s) = \frac{(-i\lambda)^2}{2} \int d^4\bar{k} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}.$$

Note, that $s = (p_1 + p_2)^2 =: p^2$. The t- and u-channel diagrams are identical, except that s will be replaced by t and u. Thus,

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2\, i\big(V(s) + V(t) + V(u)\big) - i\delta_\lambda.$$

Our renormalization condition for the full vertex (which we are computing here) requires $i\mathcal{M} = -i\lambda$ at $s = t = u = -M^2$. Thus, $-i\delta_\lambda$ needs to cancel the three V-terms for this value of the Mandelstam variables and hence we have obviously

$$-i\delta_\lambda = -(-i\lambda)^2\, 3iV(-M^2).$$

To move on, we should evaluate the integral in $V(s) \equiv V(p^2)$. In dimensional regularization, we introduce a Feynman parameter (11.2), shift the integration variable by substituting $k = l - xp$, perform a Wick rotation ($l^0 = il_E^0$) and perform the momentum integral (13.2):

$$\begin{aligned}
V(p^2) &= \frac{i}{2}\int d^d\bar{k} \frac{1}{k^2 - m^2} \frac{1}{(k+p)^2 - m^2} \\
&= \frac{i}{2}\int_0^1 dx \int d^d\bar{k} \frac{1}{\big(x((k+p)^2 - m^2) + (1-x)(k^2 - m^2)\big)^2} \\
&= \frac{i}{2}\int_0^1 dx \int d^d\bar{k} \frac{1}{(k^2 + xp^2 + 2xk\cdot p - m^2)^2} \\
&= \frac{i}{2}\int_0^1 dx \int d^d\bar{l} \frac{1}{((l - xp)^2 + xp^2 + 2x(l - xp)\cdot p - m^2)^2} \\
&= \frac{i}{2}\int_0^1 dx \int d^d\bar{l} \frac{1}{(l^2 + x(1-x)p^2 - m^2)^2} = \frac{i}{2}\int_0^1 dx \int i\, d^d\bar{l}_E \frac{1}{(-l_E^2 + x(1-x)p^2 - m^2)^2} \\
&= -\frac{1}{2}\int_0^1 dx \int d^d\bar{l}_E \frac{1}{(l_E^2 + \Delta)^2} = -\frac{1}{2}\int_0^1 dx\, \frac{\Delta^{d/2-2}}{(4\pi)^{d/2}} \frac{\Gamma(2 - d/2)}{\Gamma(2)}.
\end{aligned}$$

where $\Delta := -x(1-x)p^2 + m^2$. In the present context, we are considering massless ϕ^4 theory. Setting $m = 0$, we find

$$\delta_\lambda = (-i\lambda)^2\, 3V(-M^2) = -\frac{3(-i\lambda)^2}{2(4\pi)^{d/2}} \int_0^1 dx\, \frac{\Gamma(2 - d/2)}{(x(1-x)M^2)^{2-d/2}}.$$

Using the identities from 13.2,

$$\Gamma(2 - d/2) = \Gamma(\epsilon/2) \overset{\epsilon\to 0}{=} \frac{2}{\epsilon} - \gamma,$$

$$\Delta^{d/2-2} = \Delta^{-\epsilon/2} \overset{\epsilon\to 0}{=} 1 - \frac{\epsilon}{2}\ln\Delta, \qquad (4\pi)^{-d/2} = (4\pi)^{\epsilon/2-2} = \frac{1}{(4\pi)^2}\Big(1 + \frac{\epsilon}{2}\ln 4\pi\Big),$$

we can evaluate δ_λ in the limit $d \to 4$:

$$\begin{aligned}
\delta_\lambda &= -\frac{3(-i\lambda)^2}{2(4\pi)^2}\Big(1 + \frac{\epsilon}{2}\ln 4\pi\Big)\int_0^1 dx\, \Big(\frac{2}{\epsilon} - \gamma\Big)\Big(1 - \frac{\epsilon}{2}\ln\Delta\Big) \\
&= -\frac{3(-i\lambda)^2}{2(4\pi)^2}\int_0^1 dx\, \Big(\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln(x(1-x)M^2)\Big) \\
&= \frac{3\lambda^2}{2(4\pi)^2}\Big(\frac{2}{\epsilon} - \ln M^2 - \gamma + \ln 4\pi - \int_0^1 dx\, \ln\big(x(1-x)\big)\Big).
\end{aligned}$$

From 8.2 we know that the S matrix equals $i\mathcal{M}$ times the overall momentum conservation δ-function and the alternative LSZ reduction formula from 7.5 then tells us (recall that we consider the massless theory)

$$G^{(4)}(\{p_i\}, M, \lambda) := \int (\Pi_i d^4 x_i)\, e^{ip_1\cdot x_1} e^{ip_2\cdot x_2} e^{-ip_3\cdot x_3} e^{-ip_4\cdot x_4}\, G^{(4)}(\{x_i\}, \lambda, M) = \left(\Pi_i \frac{i}{p_i^2}\right) S$$
$$= \left(\Pi_i \frac{i}{p_i^2}\right)\left(i\mathcal{M}\,(2\pi)^4 \delta(\text{mom cons})\right),$$

where $G^{(4)}(\{p_i\}, M, \lambda)$ is the 4-point function in momentum space. We can drop the δ-function by assuming it is fulfilled[1] and are left with

$$G^{(4)}(\{p_i\}, \lambda, M) = i\mathcal{M}\, \Pi_i \frac{i}{p_i^2} = \left(-i\lambda + (-i\lambda)^2\, i\big(V(s) + V(t) + V(u)\big) - i\delta_\lambda\right) \Pi_i \frac{i}{p_i^2}.$$

For better overview, we can give this equation as

$$G^{(4)} = A(-i\lambda - \lambda^2 B - i\delta_\lambda), \qquad \delta_\lambda = C\lambda^2 \left(\frac{2}{\epsilon} - \ln M^2 + D\right)$$

with the abbreviations

$$A := \Pi_i \frac{i}{p_i^2}, \quad B := i\big(V(s) + V(t) + V(u)\big), \quad C := \frac{3}{2(4\pi)^2},$$
$$D := -\gamma + \ln 4\pi - \underbrace{\int_0^1 dx \ln\big(x(1-x)\big)}_{=-2}.$$

17.4.3 Evaluation of β

Only δ_λ depends on M, thus

$$M \frac{\partial}{\partial M} G^{(4)} = MA\left(-i \frac{\partial \delta_\lambda}{\partial M}\right) = -iMA\, C\lambda^2 \frac{\partial}{\partial M}(-2\ln M) = 2iAC\lambda^2.$$

Let's assume for now, that the first contribution to γ is at least of order λ^2, that is $\gamma = 0 + \mathcal{O}(\lambda^2)$. As we will see in (>16.4.4), this assumption is consistent. Obviously, the leading order of $G^{(4)}$ is the order λ. Thus, up to order λ^2, the term $n\gamma G^{(4)}$ is zero and the Callan-Symanzik equations reads

$$0 = \left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda}\right) G^{(4)} + \mathcal{O}(\lambda^3) = 2iAC\lambda^2 + \beta(\lambda)\frac{\partial}{\partial \lambda} G^{(4)} + \mathcal{O}(\lambda^3).$$

Since the leading order of $G^{(4)}$ is simply $G^{(4)} = -iA\lambda$, to leading order,

$$0 = 2iAC\lambda^2 - iA\beta(\lambda) + \mathcal{O}(\lambda^3) \qquad \Leftrightarrow \qquad \beta(\lambda) = 2C\lambda^2 = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3).$$

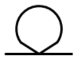
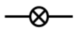
17.4.4 Renormalized 2-Point Function

If $-i\mathcal{P}^2$ is the 1PI of the scalar field, then its full propagator is

$$\frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}(-i\mathcal{P}^2)\frac{i}{p^2 - m^2} + \cdots = \frac{i}{p^2 - m^2}\sum_{n=0}^{\infty}\left((-i\mathcal{P}^2)\frac{i}{p^2 - m^2}\right)^n$$
$$= \frac{i}{p^2 - m^2}\frac{1}{1 - (-i\mathcal{P}^2)\frac{i}{p^2 - m^2}} = \frac{i}{p^2 - m^2 - \mathcal{P}^2}.$$

[1] *I don't really know, how valid this is, but in Peskin&Schröder the δ-function is suddenly gone.*

To order λ, two diagrams contribute to the 1PI:

$$-i\mathcal{P}^2(p^2) =$$  $+$  $+\,\mathcal{O}(\lambda^2)$

Using the Feynman rules from 8.1 and 15.6, in the language of mathematics, this reads

$$\begin{aligned}-i\mathcal{P}^2(p^2) &= -i\lambda\cdot\frac{1}{2}\int d^d\bar{k}\,\frac{i}{k^2-m^2} + i(p^2\delta_Z - \delta_m)\\ &= -\frac{i\lambda}{2}\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}(m^2)^{d/2-1} + i(p^2\delta_Z - \delta_m),\end{aligned}$$

where we performed a Wick rotation and used the integration formulas of section 13.2:

$$\int d^d\bar{k}\,\frac{i}{k^2-m^2} = i\int d^d\bar{k}_E\,\frac{i}{-k_E^2-m^2} = \frac{(m^2)^{d/2-1}}{(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{\Gamma(1)}.$$

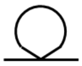
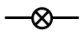
Applying the renormalization condition $\mathcal{P}^2(-M^2) = 0$ yields

$$-i\mathcal{P}^2(-M^2) = -\frac{i\lambda}{2}\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}(m^2)^{d/2-1} + i(-M^2\delta_Z - \delta_m) \overset{!}{=} 0.$$

Thus, we need $\delta_Z = 0$ and

$$-i\delta_m = \frac{i\lambda}{2}\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}(m^2)^{d/2-1}.$$

Since the first term in $-i\mathcal{P}^2(-M^2)$ is independent of p^2, this conditions also ensures $-i\mathcal{P}^2(p^2) = 0 + \mathcal{O}(\lambda^2)$ for any p^2.[1] Of course, $\mathcal{P}^2(p^2)$ can have contributions of higher orders, but in full generality we have found that

 $+$  $= 0$

Thus, the full propagator $G^{(2)}$ receives non-vanishing *corrections* only at order λ^2 and above. For the massless theory, this means

$$G^{(2)} = \frac{i}{p^2} + \mathcal{O}(\lambda^2).$$

Thus, the derivation $\partial G^{(2)}/\partial M$ is at least of order λ^2 and hence, so is the first term in the Callan-Symanzik equation. The derivation $\partial G^{(2)}/\partial\lambda$ is at least of order λ, thus the term $\beta\,\partial G^{(2)}/\lambda$ is of order λ^3 (since β is of order λ^2). Denoting only the leading orders of the terms, the Callan-Symanzik equation for the 2-point function reads

$$\mathcal{O}(\lambda^2) + \mathcal{O}(\lambda^3) + 2\gamma\,G^{(2)} = 0.$$

Solving for γ, the leading order will be λ^2.

[1] To not get confused, let's see what happens if the first term of $\mathcal{P}^2$ would depend on p^2. Then $\mathcal{P}^2$ would have the structure

$$-i\mathcal{P}^2(p^2) = f(p^2) + i(p^2\delta_Z - \delta_m).$$

The renormalization condition would give us

$$-i\mathcal{P}^2(-M^2) = f(-M^2) + i(-M^2\delta_Z - \delta_m) = 0.$$

If, for example, f is a linear function, we could not conclude $\delta_Z = 0$. In general, this condition fixes δ_Z and δ_m to be some specific functions of $-M^2$ (but *not* of p^2). And therefore, $-i\mathcal{P}^2(p^2)$ would vanish only for $p^2 = -M^2$ but not for general p^2.

17.5 General Expressions for β and γ

17.5.1 General Expression for γ

When we examined the effects of a shift $M \to M + \delta M$ in section 16.4, we found that also $\sqrt{Z}$ is shifted and we defined a shift $\delta\eta$ as follows: $\sqrt{Z} \to \sqrt{Z}(1 + \delta\eta)$. This is equivalent to

$$\sqrt{Z}' = \sqrt{Z}(1 + \delta\eta) \qquad \Leftrightarrow \qquad \delta\eta = \frac{\sqrt{Z}'}{\sqrt{Z}} - 1.$$

Since a shift in M induces a shift in $\sqrt{Z}$, Z must be a function of M and thus, $\sqrt{Z(M)}' = \sqrt{Z(M + \delta M)}$. From the definition of γ from section 16.4, we therefore find

$$\gamma = \frac{M}{\delta M}\delta\eta = \frac{M}{\delta M}\left(\frac{\sqrt{Z(M + \delta M)}}{\sqrt{Z(M)}} - 1\right) = \frac{M}{\sqrt{Z(M)}}\left(\frac{\sqrt{Z(M + \delta M)} - \sqrt{Z(M)}}{\delta M}\right) = \frac{M}{\sqrt{Z}}\frac{\partial\sqrt{Z(M)}}{\partial M}$$
$$= \frac{M}{\sqrt{Z}}\frac{1}{2\sqrt{Z}}\frac{\partial Z(M)}{\partial M} = \frac{M}{2Z}\frac{\partial Z}{\partial M}.$$

Using the definition of $\delta_Z = Z - 1$ from 15.6, we find to leading order in the coupling constant (note that δ_Z depends on λ)

$$\gamma = \frac{M}{2(1 + \delta_Z)}\frac{\partial(1 + \delta_Z)}{\partial M} = \frac{M}{2(1 + \delta_Z)}\frac{\partial\delta_Z}{\partial M} \approx \frac{M}{2}\frac{\partial\delta_Z}{\partial M}.$$

Alternatively, we can also get this formula by a more explicit analysis as follows. Since M enters only through the renormalization conditions, only the counter terms δ_X depend on M. The leading order contribution *of the counter terms* to $G^{(2)}$ is the counter propagator. In its amputated form, it simply reads $ip^2\delta_Z$, as we know from section 15.6 (there is no δ_m in a massless theory). Including the adjacent propagators, this term contributes

$$G^{(2)} = (\text{terms without counter term contributions}) + \frac{i}{p^2}\, ip^2\delta_Z\, \frac{i}{p^2}$$
$$+ (\text{higher order counter term contributions})$$
$$= (\text{terms independent of } M) + \frac{i}{p^2}\, ip^2\delta_Z\, \frac{i}{p^2} + \mathcal{O}(\lambda^3).$$

Particularly, the first appearance of δ_λ is at higher order of λ than the first appearance of δ_Z.

Now consider the Callan-Symanzik equation:

$$\left(M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + 2\gamma\right)G^{(2)} = 0.$$

The 2-point function has no term of order λ, as it can only have an even number of vertices. Thus, since we know from 16.4 that the leading order of β is λ^2, the leading order of the β-term is of order λ^3 (the order λ^0 in $G^{(2)}$ vanishes because of the derivative). Thus, to order λ^2 we can neglect the β-term and find

$$0 = M\frac{\partial}{\partial M}G^{(2)} + 2\gamma G^{(2)} = M\frac{\partial}{\partial M}\left(\frac{i}{p^2}ip^2\delta_Z\frac{i}{p^2}\right) + 2\gamma\frac{i}{p^2} \qquad \Leftrightarrow \qquad \gamma = \frac{M}{2}\frac{\partial\delta_Z}{\partial M}.$$

In (>17.4.4) we ended up with $\delta_Z = 0$. However, this calculation was only performed to order λ. In order λ^2, δ_Z will be non-zero. Thus, also the leading order of γ will be λ^2, in consistence with our result from 16.4

17.5.2 General Expression for β

When we examined the effects of a shift $M \to M + \delta M$ in section 16.4, we found that also λ is shifted and we defined a shift $\delta\lambda$ as follows: $\lambda \to \lambda + \delta\lambda$. This is equivalent to

$$\lambda' = \lambda + \delta\lambda \qquad \Leftrightarrow \qquad \delta\lambda = \lambda' - \lambda.$$

Since a shift in M induces a shift in λ, λ must be a function of M and thus, $\lambda'(M) = \lambda(M + \delta M)$. From the definition of β from section 16.4, we therefore find

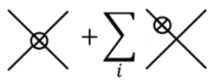
$$\beta = M\frac{\delta\lambda}{\delta M} = M\frac{\lambda' - \lambda}{\delta M} = M\frac{\lambda(M + \delta M) - \lambda(M)}{\delta M} = M\frac{\partial\lambda}{\partial M}.$$

Using the definition of $\delta_\lambda = \lambda_0 Z^2 - \lambda$ from 15.6, we find to leading order in the coupling constant

$$\begin{aligned}\beta &= M\frac{\partial\lambda}{\partial M} = M\frac{\partial}{\partial M}(\lambda_0 Z^2 - \delta_\lambda) = M\frac{\partial}{\partial M}(\lambda_0(1 + \delta_Z)^2 - \delta_\lambda) = M\frac{\partial}{\partial M}(\lambda_0(1 + 2\delta_Z) - \delta_\lambda) \\ &= M\frac{\partial}{\partial M}(-\delta_\lambda + 2\lambda_0\delta_Z) = M\frac{\partial}{\partial M}\left(-\delta_\lambda + \frac{\lambda}{2}\sum_i \delta_Z\right).\end{aligned}$$

In the last step we exchanged $\lambda_0 \to \lambda$, which is valid to leading order. Also, we introduced a sum over the external particles of the 4-point function of ϕ^4 theory. Since there is only one particle type in ϕ^4 theory, all δ_Z are equal and independent of i, thus $\sum_i \delta_Z = 4\delta_Z$. For other theories like QED, however, this formula stills holds when we sum over the external lines and use δ_{Zi} for the renormalization of the particle type of external line i.

Alternatively, we can also get this formula by a more explicit analysis as follows. To find the general expression for β, we need to start with a general expression for the M-dependent terms of $G^{(4)}$. In general, there are two contributing counter terms to order λ^2: The vertex counter term $-i\delta_\lambda$ and the propagator counter term $ip^2\delta_Z$ (recall, $m = 0 \Longrightarrow \delta_m = 0$), which can be placed at each of the four external lines (therefore the sum over the external lines i):



$$= -i\delta_\lambda + (-i\lambda)\sum_i ip_i^2\delta_Z \cdot \frac{i}{p_i^2}$$

On the right-hand side, the external propagators where neglected; we therefore need to add a factor $\Pi_i\, i/p_i^2$ to the 4-point function. Note that we found in (>17.4.4) that δ_Z has order λ term. Therefore, the second term of the expression above is of order λ^3 and can be neglected to order λ^2. However, the coincidence that the order λ of δ_Z vanishes is only true for the special case of massless ϕ^4 theory. We will continue to write this term as if δ_Z where of order λ. For final results, we can still set this order to zero. Thus,

$$G^{(4)} = \left((\text{terms independent of } M) + (-i\delta_\lambda) + (-i\lambda)\sum_i(-\delta_Z)\right)\Pi_i\frac{i}{p_i^2}$$

$$\Longrightarrow \qquad \frac{\partial G^{(4)}}{\partial M} = \left(\Pi_i\frac{i}{p_i^2}\right)\frac{\partial}{\partial M}\left(-i\delta_\lambda + i\lambda\sum_i\delta_Z\right).$$

Note, that $\sum_i 1 = 4$. The leading of $G^{(4)}$ order in λ is simply $-i\lambda\,\Pi_i\, i/p_i^2$. Thus,

$$\frac{\partial G^{(4)}}{\partial\lambda} = -i\,\Pi_i\frac{i}{p_i^2}.$$

Also, we know from (>17.5.1), that

$$\gamma = \frac{M}{2}\frac{\partial\delta_Z}{\partial M}.$$

Now, we have all terms of the Callan-Symanzik equation together and find

$$0 = \left(M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + 4\gamma\right)G^{(2)}$$
$$= \left(M\left(\Pi_i\frac{i}{p_i^2}\right)\frac{\partial}{\partial M}\left(-i\delta_\lambda + i\lambda\sum_i\delta_Z\right)\right) + \left(\beta\left(-i\,\Pi_i\frac{i}{p_i^2}\right)\right) + \left(4\frac{M}{2}\frac{\partial\delta_Z}{\partial M}\left(-i\lambda\,\Pi_i\frac{i}{p_i^2}\right)\right)$$

$$\Leftrightarrow \qquad 0 = \left(M\frac{\partial}{\partial M}\left(\delta_\lambda - \lambda\sum_i\delta_Z\right)\right) + \beta + \left(\lambda\frac{M}{2}\frac{\partial}{\partial M}\sum_i\delta_Z\right)$$

$$\Leftrightarrow \qquad \beta = -M\frac{\partial\delta_\lambda}{\partial M} + \frac{\lambda M}{2}\frac{\partial}{\partial M}\sum_i\delta_Z = M\frac{\partial}{\partial M}\left(-\delta_\lambda + \frac{\lambda}{2}\sum_i\delta_Z\right).$$

In the present case, where all the external particles are equal, also all the δ_Z are equal and we can simply replace $\sum_i = 4$. However, for QED, externals particles can be electrons or photons and then δ_Z depends on i.

Taking $\delta_\lambda = -C\lambda^2\ln M^2 + (\text{term independent of } M)$ from (>17.4.2), where $2C = 3/(4\pi)^2$, this formula gives us for ϕ^4 theory (where $\delta_Z = 0$ to order λ)

$$\beta = M\frac{\partial}{\partial M}C\lambda^2\ln M^2 = 2C\lambda^2,$$

just as we found in section 16.4.

17.6 Callan-Symanzik Equation for QED

17.6.1 Derivation of the Callan-Symanzik Equation for QED

We could derive the Callan-Symanzik equation for QED in the same way as for ϕ^4 theory. However, it is useful, to go an alternative, if still similar, way. Let's again start with an (n_2, n_3)-point function of n_2 fermions and n_3 photons. It is connected to the bare (n_2, n_3)-point function by

$$G^{(n_2,n_3)}(\{x_i\}, g, m, \mu) = Z_2^{-n_2/2}Z_3^{-n_3/2}G_0^{(n_2,n_3)}(\{x_i\}, g_0, m_0),$$

where $g = e$ is the (renormalized) coupling constant, m the (renormalized) fermion mass and g_0, m_0 the bare quantities. From the definition for δ_1 in section 15.3, we see that

$$Z_1 = Z_2\sqrt{Z_3}\frac{g_0}{g} \qquad \Leftrightarrow \qquad g = \frac{Z_2\sqrt{Z_3}}{Z_1}g_0$$

(here, $\sqrt{Z_3} \coloneqq \sqrt{Z_3(0)}$). We know from the end of section 13.2, that g_0 has mass dimension $(4-d)/2$. To keep g dimensionless in any dimension, we modify the renormalization for the present purpose a little bit:

$$g = Z^{-1}\mu^{(d-4)/2}g_0, \qquad Z \coloneqq \frac{Z_1}{Z_2\sqrt{Z_3}},$$

where μ is some mass scale. Note, that in the case of $d = 4$, the two renormalization rules totally coincide. We take μ as exactly the μ from section 13.2. This μ appears in computations of dimensional regularization and thus it occurs in the regularization parameters δ and hence also in Z_i and the renormalized mass m. On the other hand, $G_0^{(n_2,n_3)}$ is μ independent, and therefore, we can write

$$0 = \mu\frac{d}{d\mu}G_0^{(n_2,n_3)} = \mu\frac{d}{d\mu}\left(Z_2^{n_2/2}Z_3^{n_3/2}G^{(n_2,n_3)}\right).$$

Consider

$$\mu\frac{d}{d\mu}Z_i^{n_i/2} = \mu\frac{n_i}{2}Z_i^{n_i/2-1}\frac{dZ_i}{d\mu} = Z_i^{n_i/2}\frac{n}{2}\frac{\mu}{Z_i}\frac{dZ_i}{d\mu},$$

$$\mu\frac{d}{d\mu}G^{(n_2,n_3)} = \mu\frac{\partial}{\partial\mu}G^{(n_2,n_3)} + \mu\frac{\partial G^{(n_2,n_3)}}{\partial g}\frac{dg}{d\mu} + \mu\frac{\partial G^{(n_2,n_3)}}{\partial m}\frac{dm}{d\mu}.$$

Hence,

$$0 = Z_2^{n_2/2}Z_3^{n_3/2}\left(\mu\frac{\partial}{\partial\mu} + \frac{n_2}{2}\frac{\mu}{Z_2}\frac{dZ_2}{d\mu} + \frac{n_3}{2}\frac{\mu}{Z_3}\frac{dZ_3}{d\mu} + \mu\frac{dg}{d\mu}\frac{\partial}{\partial g} + \mu\frac{dm}{d\mu}\frac{\partial}{\partial m}\right)G^{(n_2,n_3)}.$$

Defining

$$\gamma_{2,3} \coloneqq \frac{1}{2}\frac{\mu}{Z_{2,3}}\frac{dZ_{2,3}}{d\mu}, \qquad \gamma_m \coloneqq \frac{\mu}{m}\frac{dm}{d\mu}, \qquad \beta \coloneqq -\mu\frac{dg}{d\mu}$$

we arrive at the Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + n_2\gamma_2 + n_3\gamma_3 + \beta\frac{\partial}{\partial g} + m\gamma_m\frac{\partial}{\partial m}\right)G^{(n_2,n_3)} = 0.$$

17.6.2 General Expression for β and γ

Using the relation between g and g_0 from (>17.6.1) as well as the fact that g_0 is independent of μ, we find

$$0 = \mu\frac{d}{d\mu}g_0 = \mu\frac{d}{d\mu}\left(Z\mu^{(4-d)/2}g\right) = \mu\left(\frac{dZ}{d\mu}\mu^{(4-d)/2}g + Z\frac{d\mu^{(4-d)/2}}{d\mu}g + Z\mu^{(4-d)/2}\frac{dg}{d\mu}\right).$$

Solving this equation for $\beta \coloneqq -\mu\, dg/d\mu$, we find

$$\beta = -\mu\frac{dg}{d\mu} = \frac{\mu g}{Z\mu^{(4-d)/2}}\left(\frac{dZ}{d\mu}\mu^{(4-d)/2} + Z\frac{d\mu^{(4-d)/2}}{d\mu}\right) = g\left(\frac{\mu}{Z}\frac{dZ}{d\mu} + \frac{4-d}{2}\right).$$

Expanding Z^{-1}, using its definition from (>17.6.1), we find

$$\begin{aligned} Z \coloneqq \frac{Z_1}{Z_2\sqrt{Z_3}} &= \frac{1+\delta_1}{(1+\delta_2)\sqrt{1+\delta_3}} = (1+\delta_1)\big(1-\delta_2+\mathcal{O}(g^3)\big)\left(1-\frac{1}{2}\delta_3+\mathcal{O}(g^3)\right) \\ &= 1+\delta_1-\delta_2-\frac{1}{2}\delta_3+\mathcal{O}(g^3). \end{aligned}$$

Note, that δ_i is of order g^2. Thus, using $4-d=\epsilon$,

$$\beta = g\left(\frac{\mu}{Z}\frac{d}{d\mu}\left(\delta_1-\delta_2-\frac{1}{2}\delta_3\right) + \frac{4-d}{2} + \mathcal{O}(g^3)\right) = g\left(\frac{\epsilon}{2} + \mu\frac{d}{d\mu}\left(\delta_1-\delta_2-\frac{1}{2}\delta_3\right)\right) + \mathcal{O}(g^4)$$

To the given order, we could neglect $Z = 1+\mathcal{O}(g^2)$. Usually (if β is not multiplied by a term $\sim 1/\epsilon$), we can also neglect the first term, leaving us with

$$\beta = g\mu\frac{d}{d\mu}\left(\delta_1-\delta_2-\frac{1}{2}\delta_3\right) + \mathcal{O}(g^4).$$

The $\gamma_{2,3}$ can obviously be given as

$$\gamma_{2,3} = \frac{1}{2}\frac{\mu}{Z_{2,3}}\frac{dZ_{2,3}}{d\mu} = \frac{1}{2}\frac{\mu}{1+\delta_{2,3}}\frac{d\big(1+\delta_{2,3}\big)}{d\mu} = \frac{\mu}{2}\frac{d\delta_{2,3}}{d\mu} + \mathcal{O}(g^4)$$

(γ_m is not given here yet).

17.6.3 Results for β and γ

In (>16.4.2) we found ($g = e$)

$$\delta_1 = \delta_2 = -\frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right),$$

$$\delta_3 = -\frac{4}{3}\frac{g^2}{(4\pi)^2}\left(\frac{2}{\epsilon} + \text{finite}\right).$$

(We can choose a subtraction scheme, where the finite terms are dropped; then still all infinities are subtracted by the δ's).

To leading order, we can set $Z = 1$ to find (using $\epsilon = 4 - d$)

$$\mu\frac{dg}{d\mu} = \mu\frac{d}{d\mu}\left(g_0 Z^{-1}\mu^{(d-4)/2}\right) = \mu\frac{d}{d\mu}\left(g_0\mu^{(d-4)/2}\right) + \mathcal{O}(g^3) = \frac{d-4}{2}\underbrace{g_0\mu^{(d-4)/2}}_{=g+\mathcal{O}(g^3)}$$
$$= -\frac{\epsilon}{2}g + \mathcal{O}(g^3).$$

Thus, $\mu\, dg^2/d\mu = -\epsilon g^2 + \mathcal{O}(g^4)$. Now, recall $Z = 1 + \delta_1 - \delta_2 - \delta_3/2 + \mathcal{O}(g^3) = 1 - \delta_3/2 + \mathcal{O}(g^3)$ from (>17.6.2). Thus,

$$\mu\frac{dZ^{-1}}{d\mu} = \mu\frac{d}{d\mu}\frac{1}{1-\frac{1}{2}\delta_3} + \mathcal{O}(g^3) = \mu\frac{d}{d\mu}\left(1 + \frac{1}{2}\delta_3\right) + \mathcal{O}(g^3) = \frac{\mu}{2}\frac{d\delta_3}{d\mu} + \mathcal{O}(g^3)$$
$$= \left(-\frac{4}{3}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\right)\frac{\mu}{2}\frac{dg^2}{d\mu} + \mathcal{O}(g^3) = \left(-\frac{4}{3}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\right)\frac{-\epsilon g^2}{2} + \mathcal{O}(g^3) = \frac{g^2}{12\pi^2} + \mathcal{O}(g^3).$$

Finally, we can compute the β function to order g^3:

$$\beta = -\mu\frac{dg}{d\mu} = -\mu\frac{d}{d\mu}\left(g_0 Z^{-1}\mu^{(d-4)/2}\right) = -g_0 Z^{-1}\left(\mu\frac{d\mu^{(d-4)/2}}{d\mu}\right) - g_0\left(\mu\frac{dZ^{-1}}{d\mu}\right)\mu^{(d-4)/2}$$
$$= -g_0 Z^{-1}\left(-\frac{\epsilon}{2}\mu^{(d-4)/2}\right) - \underbrace{g_0\mu^{(d-4)/2}}_{=Zg}\left(\frac{g^2}{12\pi^2} + \mathcal{O}(g^3)\right) = \frac{\epsilon}{2}g - Z\frac{g^3}{12\pi^2} + \mathcal{O}(g^4)$$
$$= \frac{\epsilon}{2}g - \frac{g^3}{12\pi^2} + \mathcal{O}(g^4).$$

Using $\mu\, dg^2/d\mu = -\epsilon g^2 + \mathcal{O}(g^4)$, we find

$$\gamma_2 = \frac{\mu}{2}\frac{d\delta_2}{d\mu} = -\frac{1}{8\pi^2\epsilon}\frac{\mu}{2}\frac{dg^2}{d\mu} = -\frac{1}{8\pi^2\epsilon}\frac{-\epsilon g^2}{2} + \mathcal{O}(g^4) = \frac{g^2}{16\pi^2} + \mathcal{O}(g^4),$$

$$\gamma_3 = \frac{\mu}{2}\frac{d\delta_3}{d\mu} = -\frac{4}{3}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\frac{\mu}{2}\frac{dg^2}{d\mu} = -\frac{4}{3}\frac{1}{(4\pi)^2}\frac{2}{\epsilon}\frac{-\epsilon g^2}{2} + \mathcal{O}(g^4) = \frac{g^2}{12\pi^2} + \mathcal{O}(g^4).$$

17.7 Evolution of Coupling Constants

17.7.1 New Form of the Callan-Symanzik Equation

The leading order of the 2-point function of ϕ^4 theory reads $G^{(2)}(p^2) = i/p^2 + \cdots$; thus, its mass dimension is -2. Therefore, we can write it in the form

$$G^{(2)}(p^2) = \frac{i}{p^2}f(-p^2/m^2).$$

Let's use the variable $\wp := \sqrt{-p^2}$ instead of p. That is, $\wp^2 = -p^2$, but $\wp$ is a scalar and p a four-vector. We then write $G^{(2)}(\wp)$ instead of $G^{(2)}(p^2)$:

$$G^{(2)}(\wp) = \frac{i}{p^2} f(-p^2/m^2) = \frac{i}{\wp^2} f(-\wp^2/m^2).$$

We have absorbed minus sign from the substitution $p^2 = -\wp^2$ into the unknown function f. Let's see, what a derivative with respect to $\wp$ instead of M would give us:

$$\wp \frac{\partial}{\partial \wp} G^{(2)}(\wp) = \wp \frac{\partial}{\partial \wp}\left(\frac{i}{\wp^2} f(-\wp^2/M^2)\right) = \wp\left(\frac{-2i}{\wp^3}\right) f(-\wp^2/M^2) + \wp \frac{i}{\wp^2}\left(\frac{-2\wp}{M^2} f'(-\wp^2/M^2)\right)$$
$$= -2G^{(2)}(\wp) - \frac{2i}{M^2} f'(-\wp^2/M^2)$$
$$\Leftrightarrow \qquad \frac{2i}{M^2} f'(-\wp^2/M^2) = -2G^{(2)}(\wp) - \wp \frac{\partial}{\partial \wp} G^{(2)}(\wp).$$

Thus, we can write the derivative with respect to M as

$$M \frac{\partial}{\partial M} G^{(2)}(\wp) = M \frac{\partial}{\partial M} \frac{i}{\wp^2} f(-\wp^2/M^2) = M \frac{i}{\wp^2} f'(-\wp^2/M^2) \frac{2\wp^2}{M^3} = \frac{2i}{M^2} f'(-\wp^2/M^2)$$
$$= -2G^{(2)}(\wp) - \wp \frac{\partial}{\partial \wp} G^{(2)}(\wp).$$

and the Callan-Symanzik equation for the 2-point function as

$$0 = \left(M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 2\gamma(\lambda)\right) G^{(2)}(\wp) = \left(-2 - \wp \frac{\partial}{\partial \wp} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 2\gamma(\lambda)\right) G^{(2)}(\wp).$$

17.7.2 General Solution of the Callan-Symanzik Equation

To find the general solution of the Callan-Symanzik equation, let's start by simplifying it. Writing the momentum derivative as

$$\wp \frac{\partial}{\partial \wp} = (\wp/M) \frac{\partial}{\partial(\wp/M)} = \frac{\partial}{\partial \ln(\wp/M)},$$

the Callan-Symanzik equation reads

$$\left(\frac{\partial}{\partial \ln(\wp/M)} - \beta(\lambda) \frac{\partial}{\partial \lambda} - 2\gamma(\lambda) + 2\right) G^{(2)}(\wp) = 0.$$

Now let's, for simplicity, substitute $t \coloneqq \ln \wp/M$ and $v \coloneqq -\beta$ and $\rho \coloneqq 2\gamma - 2$ as well as $x \coloneqq \lambda$. Then, we can also replace $G^{(2)}(\wp, \lambda)$ by $D(t, x)$:

$$\left(\frac{\partial}{\partial t} + v(x) \frac{\partial}{\partial x} - \rho(x)\right) D(t, x) = 0 \qquad \Leftrightarrow \qquad \frac{\partial}{\partial t} D(t, x) = \rho(x) D(t, x) - v(x) \frac{\partial}{\partial x} D(t, x).$$

This equation has a beautiful hydrodynamic-bacteriological analogy: It describes the density of bacteria $D(t, x)$ at time t and position x along a one-dimensional tube. The tube is filled with water that flows with velocity $v(x)$. Above the pipe, heat/light sources are placed at different positions, so that the growth rate of the bacteria ρ is position dependent.

Consider an arbitrary position x_0. The change in bacteria density $\partial D/\partial t|_{x_0}$ at this position equals the growth of the bacteria number $\rho D|_{x_0}$ on this position. But also, since the bacteria are carried by the flowing water through the pipe, the change $\partial D/\partial t|_{x_0}$ will also depend on the density at neighbouring positions. Let's assume $v(x_0) > 0$, such that the flow is directed to higher x. If $\partial D/\partial x|_{x_0}$ is positive, on the left-hand side of x_0, there are less bacteria than at x_0. Thus, we need to subtract $v\, \partial D/\partial x$ from the change, to take the flow into account.

Take an element of the fluid, which is inhabited by a certain number of bacteria. Let this element be at position x at a certain time t. Let then $\bar{x}(t,x)$ be the position, where this element has been at $t=0$. Note that the larger t is the smaller is $\bar{x}$, as well as $\bar{x}(0,x)$. We then integrate back in time to find $\bar{x}(t,x)$:

$$\bar{x}(t,x)=\int_t^0 dt'\, v\big(\bar{x}(t',x)\big) \qquad \Leftrightarrow \qquad \frac{\partial}{\partial t}\bar{x}(t,x)=-v\big(\bar{x}(t,x)\big), \qquad \text{with} \qquad \bar{x}(0,x)=x.$$

By separation of variables, we know that

$$\frac{dy(x)}{dx}=f\big(y(x)\big) \qquad \Leftrightarrow \qquad dx=\frac{dy}{f(y)} \qquad \Leftrightarrow \qquad \int_{x_1}^{x_2} dx=\int_{y(x_1)}^{y(x_2)} dy\,\frac{1}{f(y)}.$$

In our case, this means

$$\frac{\partial}{\partial t}\bar{x}(t,x)=-v\big(\bar{x}(t,x)\big) \qquad \Leftrightarrow \qquad \int_0^t dt' = -\int_{\bar{x}(0,x)}^{\bar{x}(t,x)} d\bar{x}'\,\frac{1}{v(\bar{x}')}.$$

If we differentiate this equation by x, the left-hand side vanishes and for the right-hand ween need Leibniz's rule (recall $\bar{x}(0,x)=x$):

$$0=-\frac{d}{dx}\int_x^{\bar{x}(t,x)} d\bar{x}'\,\frac{1}{v(\bar{x}')}=-\left(\frac{1}{v\big(\bar{x}(t,x)\big)}\frac{\partial\bar{x}(t,x)}{\partial x}-\frac{1}{v(x)}\right) \qquad \Leftrightarrow \qquad \frac{\partial\bar{x}(t,x)}{\partial x}=\frac{v\big(\bar{x}(t,x)\big)}{v(x)}.$$

Using these relations, we can show that the solution to the differential equation above reads

$$D(t,x)=D_0\big(\bar{x}(t,x)\big)\,\exp\left(\int_0^t dt'\,\rho\big(\bar{x}(t',x)\big)\right).$$

Let's check that this is true, by first computing the following two derivatives:

$$\frac{\partial}{\partial t}D(t,x)=\frac{\partial D_0\big(\bar{x}(t,x)\big)}{\partial t}\,\exp\left(\int_0^t dt'\,\rho\big(\bar{x}(t',x)\big)\right)+D(t,x)\,\rho\big(\bar{x}(t,x)\big),$$

$$\frac{\partial}{\partial x}D(t,x)=\frac{\partial D_0\big(\bar{x}(t,x)\big)}{\partial x}\,\exp\left(\int_0^t dt'\,\rho\big(\bar{x}(t',x)\big)\right)+D(x,t)\int_0^t dt'\,\frac{\partial}{\partial x}\rho\big(\bar{x}(t',x)\big).$$

Consider

$$\frac{\partial D_0\big(\bar{x}(t,x)\big)}{\partial t}=D_0'\big(\bar{x}(t,x)\big)\frac{\partial\bar{x}(t,x)}{\partial t}=-D_0'\big(\bar{x}(t,x)\big)v\big(\bar{x}(t,x)\big),$$

$$\frac{\partial D_0\big(\bar{x}(t,x)\big)}{\partial x}=D_0'\big(\bar{x}(t,x)\big)\frac{\partial\bar{x}(t,x)}{\partial x}=D_0'\big(\bar{x}(t,x)\big)\frac{v\big(\bar{x}(t,x)\big)}{v(x)},$$

$$\begin{aligned}\int_0^t dt'\,\frac{\partial}{\partial x}\rho\big(\bar{x}(t',x)\big)&=\int_0^t dt'\frac{\rho\big(\bar{x}(t',x)\big)}{\partial\bar{x}(t',x)}\frac{\partial\bar{x}(t',x)}{\partial x}=\int_0^t dt'\,\frac{\rho\big(\bar{x}(t',x)\big)}{\partial\bar{x}(t',x)}\frac{v\big(\bar{x}(t',x)\big)}{v(x)}\\&=-\frac{1}{v(x)}\int_0^t dt'\,\frac{\rho\big(\bar{x}(t',x)\big)}{\partial\bar{x}(t',x)}\frac{\partial\bar{x}(t',x)}{\partial t'}=-\frac{1}{v(x)}\Big(\rho\big(\bar{x}(t,x)\big)-\rho\big(\bar{x}(0,x)\big)\Big)\\&=-\frac{1}{v(x)}\Big(\rho\big(\bar{x}(t,x)\big)-\rho(x)\Big).\end{aligned}$$

Let's write all those terms as short as possible and plug them into the differential equation:

$$
\begin{aligned}
&\frac{\partial}{\partial t} D-\rho(x)\, D+v(x)\, \frac{\partial}{\partial x} D \\
&\quad =\frac{\partial D_0}{\partial t} \exp(\cdots)+D\, \rho(\bar{x})-\rho(x)\, D+v(x)\left(\frac{\partial D_0}{\partial x} \exp(\cdots)+D \int_0^t d t' \,\frac{\partial \rho(\bar{x})}{\partial x}\right) \\
&\quad =-D_0'\, v(\bar{x}) \exp(\cdots)+D\, \rho(\bar{x})-\rho(x)\, D \\
&\qquad\qquad\qquad\qquad +v(x)\left(D_0'(\bar{x}) \frac{v(\bar{x})}{v(x)} \exp(\cdots)-D \frac{1}{v(x)}(\rho(\bar{x})-\rho(x))\right) \\
&\quad =-D_0'\, v(\bar{x}) \exp(\cdots)+D\, \rho(\bar{x})-\rho(x)\, D+D_0'(\bar{x})\, v(\bar{x}) \exp(\cdots)-D(\rho(\bar{x})-\rho(x))=0 .
\end{aligned}
$$

Now, that we have shown that the $D(x, t)$ given above is indeed a solution, we can translate it back to our field theory problem:

$$
D(t, x)=G^{(2)}(\wp, \lambda), \qquad D_0(\bar{x}(t, x))=G_0\left(\bar{\lambda}(\wp, \lambda)\right), \qquad t=\ln \frac{\wp}{M},
$$

$$
v=-\beta, \qquad \rho=2 \gamma-2, \qquad x=\lambda .
$$

Thus, $t=0$ corresponds to $\wp=M$. The translation yields

$$
\begin{aligned}
D(t, x) & =D_0(\bar{x}(t, x)) \, \exp \left(\int_0^t d t' \,\rho\left(\bar{x}\left(t', x\right)\right)\right) \\
& =G^{(2)}(\wp, \lambda)=G_0\left(\bar{\lambda}(\wp, \lambda)\right) \, \exp \left(\int_{\wp'=M}^{\wp'=\wp} \mathrm{d} \ln \left(\wp'/M\right) \,\left(2 \gamma\left(\bar{\lambda}\left(\wp', \lambda\right)-2\right)\right)\right) .
\end{aligned}
$$

The other formulas, we came across, translate as follows:

Bacteria in a Tube	Quantum Field Theory
$\frac{\partial}{\partial t} \bar{x}(t, x)=-v(\bar{x}(t, x))$	$\frac{\partial}{\partial \ln (\wp / M)} \bar{\lambda}(\wp, \lambda)=\beta\left(\bar{\lambda}(\wp, \lambda)\right)$
$\bar{x}(0, x)=x$	$\bar{\lambda}(\wp=M, \lambda)=\lambda$
$\int_0^t d t'=-\int_x^{\bar{x}(t, x)} d \bar{x}' \frac{1}{v\left(\bar{x}'\right)}$	$\int_{\wp'=M}^{\wp'=\wp} \mathrm{d} \ln \left(\wp'/M\right)=\int_\lambda^{\bar{\lambda}(\wp, \lambda)} d \bar{\lambda}' \frac{1}{\beta\left(\bar{\lambda}'\right)}$
$\frac{\partial \bar{x}(t, x)}{\partial x}=\frac{v(\bar{x}(t, x))}{v(x)}$	$\frac{\partial \bar{\lambda}(\wp, \lambda)}{\partial \lambda}=\frac{\beta\left(\bar{\lambda}(\wp, \lambda)\right)}{\beta(\lambda)}$

Plugging in the first of these four equation, we can bring the last one into the form

$$
\begin{aligned}
0= & \;\beta\left(\bar{\lambda}(\wp, \lambda)\right)-\beta(\lambda) \frac{\partial \bar{\lambda}(\wp, \lambda)}{\partial \lambda}=\left(\frac{\partial}{\partial \ln (\wp / M)}-\beta(\lambda) \frac{\partial}{\partial \lambda}\right) \bar{\lambda}(\wp, \lambda) \\
& =\left(\frac{\wp}{M} \frac{\partial}{\partial(\wp / M)}-\beta(\lambda) \frac{\partial}{\partial \lambda}\right) \bar{\lambda}(\wp, \lambda)=\left(\wp \frac{\partial}{\partial \wp}-\beta(\lambda) \frac{\partial}{\partial \lambda}\right) \bar{\lambda}(\wp, \lambda) .
\end{aligned}
$$

Finally, let's bring our main result into a more convenient form. One term in the integral in the exponent of the general solution formula for $G^{(2)}$ is simply a constant, namely -2. This term yields

$$
\exp \left(\int_{\wp'=M}^{\wp'=\wp} d l n\left(\wp'/M\right) \,(-2)\right)=\exp (-2(l n(\wp / M)-l n(M / M)))=\exp (-2\, l n(\wp / M))=\frac{M^2}{\wp^2} .
$$

Redefining G_0 (absorbing M^2 and spitting out a factor i), we can write the result as

$$
G^{(2)}(\wp, \lambda)=\frac{i}{\wp^2} G_0\left(\bar{\lambda}(\wp, \lambda)\right) \, \exp \left(2 \int_{\wp'=M}^{\wp'=\wp} \mathrm{d} \ln \left(\wp'/M\right) \,\gamma\left(\bar{\lambda}\left(\wp', \lambda\right)\right)\right) .
$$

17.8 The Running Coupling

17.8.1 Leading Order Solution for the Running Coupling of ϕ^4 Theory

To solve the differential equation for $\bar{\lambda}$, it is actually easier to use directly its integral form

$$\int_{\wp'=M}^{\wp'=\wp} d\ln(\wp'/M) = \int_{\lambda}^{\bar{\lambda}(\wp,\lambda)} d\bar{\lambda}' \frac{1}{\beta(\bar{\lambda}')},$$

derived in (>17.7.2). Using $\beta(\lambda) = 3\lambda^2/(4\pi)^2$ from section 16.4, this equation yields

$$\int_{\wp'=M}^{\wp'=\wp} d\ln(\wp'/M) = \frac{(4\pi)^2}{3}\int_{\lambda}^{\bar{\lambda}(\wp,\lambda)} d\bar{\lambda}' \frac{1}{\bar{\lambda}'^2}$$

$$\Leftrightarrow \qquad \ln(\wp/M) = \frac{(4\pi)^2}{3}\left[-\frac{1}{\bar{\lambda}'}\right]_{\lambda}^{\bar{\lambda}(\wp,\lambda)}$$

$$\Leftrightarrow \qquad \bar{\lambda}(\wp,\lambda) = \frac{\lambda}{1 - \frac{3\lambda}{(4\pi)^2}\ln(\wp/M)}.$$

17.8.2 Equality to Wilson's Approach

In section 16.3, we found

$$\lambda' = \lambda - \frac{3\lambda^2}{(4\pi)^2}\ln 1/b,$$

where we integrated out all scales from a cutoff Λ to a scale $b\Lambda$ with $0 < b < 1$. Our current result reads

$$\bar{\lambda}(\wp,\lambda) = \frac{\lambda}{1 - \frac{3\lambda}{(4\pi)^2}\ln(\wp/M)} \approx \lambda + \frac{3\lambda^2}{(4\pi)^2}\ln(\wp/M) + \mathcal{O}(\lambda^3)$$
$$= \lambda - \frac{3\lambda^2}{(4\pi)^2}\ln((\wp/M)^{-1}) + \mathcal{O}(\lambda^3).$$

Obviously, $\bar{\lambda} = \lambda'$, if

$$\frac{\wp}{M} = b = \frac{b\Lambda}{\Lambda}.$$

The correspondence of $\wp \leftrightarrow b\Lambda$ and $M \leftrightarrow \Lambda$ is quite clear: We have defined the theory at scale M/with cutoff Λ, but are interested in scale $\wp$/scale $b\Lambda$.

17.8.3 Leading Order Solution for the Running Coupling of QED

The Fourier transform of the Coulomb potential reads

$$V(\vec{q}) = -\frac{e^2(q^2)}{|\vec{q}|^2} = -\frac{e_0^2}{|\vec{q}|^2}\frac{1}{1-\Pi(q^2)},$$

as we know from section 13.5. We also plugged in the formula for $e^2(q^2)$ from section 13.1.

In section 13.1, we found the full photon propagator – which is the 2-point functions with two photon fields. It read

$$G_{\mu\nu}^{(2)} = \frac{-i\eta_{\mu\nu}}{q^2}\frac{1}{1-\Pi(q^2)} = \frac{i\eta_{\mu\nu}}{q^2}\frac{|\vec{q}|^2}{e_0^2}V(\vec{q}) = \frac{i\eta_{\mu\nu}}{e_0^2}\frac{|\vec{q}|}{q^2}V(\vec{q}) \qquad \Leftrightarrow \qquad \frac{i\eta_{\mu\nu}}{e_0^2}V(\vec{q}) = \frac{q^2}{|\vec{q}|^2}G_{\mu\nu}^{(2)}.$$

We consider the massless limit of QED and specify a scale M, at which we "define the theory", that is, at which the renormalization conditions are imposed to hold. That is, $e_r = e(q^2 = M^2)$.

The 2-point photon function contains only two photon fields and no fermion fields; that is, $m = 2$ and $n = 0$ in the Callan-Symanzik equation for QED in section 16.6, which gives exactly the Callan-Symanzik equation for the 2-point function of a massless scalar field. Defining $q\!\!\!/ \coloneqq \sqrt{-q^2}$, such that $q\!\!\!/^2 = -q^2$ (but $q\!\!\!/$ being a scalar, not a four-vector) and changing the scale derivative into a momentum derivative as in (>17.7.1), the Callan-Symanzik equation reads (note that $\partial/\partial\lambda = -\partial/\partial e$)[1]

$$\left(q\!\!\!/ \frac{\partial}{\partial q\!\!\!/} + \beta(e_r)\frac{\partial}{\partial e_r} - 2\gamma_3 + 2\right) G_{\mu\nu}^{(2)} = 0.$$

According to section 16.7, it has the solution

$$G_{\mu\nu}^{(2)}(q\!\!\!/, e_r) = \frac{1}{q\!\!\!/^2} G_{0\mu\nu}(\bar{e}_r)\exp(\cdots).$$

At $q\!\!\!/ = M$, we know that $e(q^2) = e_r$ at $q^2 = M^2$. Thus, at $q\!\!\!/ = M$,

$$V(\vec{q}) = -\frac{e_r^2}{|\vec{q}|^2}.$$

At $q\!\!\!/ = M$, the integral in the exponent vanishes automatically. Also, $\bar{e}_r = e_r$ at $q\!\!\!/ = M$. Thus, at $q\!\!\!/ = M$, our general solution reads

$$G_{\mu\nu}^{(2)}(q\!\!\!/ = M, e_r) = \frac{1}{-q^2} G_{0\mu\nu}(e_r) \stackrel{!}{=} \frac{i\eta_{\mu\nu}}{e_0^2}\frac{|\vec{q}|}{q^2}\left(-\frac{e_r^2}{|\vec{q}|^2}\right) \qquad \Leftrightarrow \qquad G_{0\mu\nu}(e_r) = \frac{i\eta_{\mu\nu}}{q^2}\frac{e_r^2}{e_0^2}.$$

Also, the running charge obeys ($d\lambda = -de_r$)

$$\int_{q\!\!\!/'=M}^{q\!\!\!/'=q\!\!\!/} d\ln(q\!\!\!/'/M) = -\int_{e_r}^{\bar{e}_r(q\!\!\!/,e_r)} de_r' \frac{1}{\beta(e_r')},$$

just as in section 16.7. The β-function is known from section 16.6 to be $\beta = -e^3/12\pi^2$. Thus, the equation above yields

$$\ln(q\!\!\!/ /M) = 12\pi^2 \int_{e_r}^{\bar{e}_r(q\!\!\!/,e_r)} de_r' \frac{1}{e_r'^3} = 12\pi^2 \left[\frac{-1}{2e_r'^2}\right]_{e_r}^{\bar{e}_r(q\!\!\!/,e_r)}$$

$$\Leftrightarrow \qquad \bar{e}_r^2(q\!\!\!/, e_r) = \frac{e_r^2}{1 - \frac{e_r^2}{6\pi^2}\ln(q\!\!\!/ /M)}.$$

[1] For some reason that I do not understand, in this case γ is zero. However, since $\gamma_3 \sim \partial\delta_3/\partial M$ and $\delta_3 = \mathcal{O}(e^2)$ (>16.4.2), this γ_3 would contribute only to higher orders of e than we are interested in. Thus, it is irrelevant for us, if it vanishes or not.

18 Non-Abelian Gauge Theories

18.1 Feynman Rules

18.1.1 The Yang-Mills Lagrangian

In section 9.3, we found the Yang-Mills Lagrangian

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(i\not{D} - m)\Psi,$$

where $\Psi = (\psi_1, \dots, \psi_N)^T$. Using as $F_{\mu\nu} = F_{\mu\nu}^a t_a$, where the generators t_a are normalized by $\operatorname{Tr} t_a t_b = 1/2$, we find (see section 3.7)

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \overline{\Psi}(i\not{D} - m)\Psi.$$

Here,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc}\, A_\mu^b A_\nu^c, \qquad D_\mu = \partial_\mu + ig\, A_\mu^a t^a.$$

Recall that $a = 1, \dots, N^2 - 1$ and $t_a \in \mathbb{C}^{N\times N}$.

18.1.2 The Fermion Propagator

The part of the Yang-Mills Lagrangian containing fermion fields reads

$$\overline{\Psi}(i\not{D} - m)\Psi, \qquad D_\mu = \partial_\mu + ig\, A_\mu^a t^a.$$

Since t^a is a matrix, unit matrix is implied for the first term of D_μ, that is for the ∂_μ, and also for the term containing the mass m. Let's denote this unit matrix explicitly as $\mathbb{I}_S$, since it is the unit matrix in the space of the symmetry group.

On the other hand, $\not{D}$ is also a matrix in Dirac space due to the γ matrix. Thus, the mass term also has a unit matrix $\mathbb{I}_D$ of Dirac space implied, which has dimensionality four in the case of $d = 4$.[1]

We want to write the expression above in index notation. Using $\alpha, \beta, \dots = 1, \dots, 4$ as Dirac indices and $i, j, \dots = 1, \dots, N$ as symmetry group indices, it reads

$$\overline{\Psi}_{i\alpha}(i\not{D} - m)_{ij\alpha\beta}\Psi_{j\beta} = \overline{\Psi}_{i\alpha}\big(i\not{D}_{ij\alpha\beta} - m\delta_{ij}\delta_{\alpha\beta}\big)\Psi_{j\beta},$$

where

$$\not{D}_{ij\alpha\beta} = \gamma_{\alpha\beta}^\mu D_{\mu,ij} = \gamma_{\alpha\beta}^\mu\big(\partial_\mu\delta_{ij} + igA_\mu^a t_{ij}^a\big).$$

Note that Ψ is a vector of vectors:

$$\Psi = (\psi_1, \dots, \psi_N)^T,$$

where each ψ_i is a Dirac vector (spinor).

If we consider the *free* field terms of the fermions only, this expression simplifies to

$$\overline{\Psi}_{i\alpha}\left(i\gamma_{\alpha\beta}^\mu\partial_\mu\delta_{ij} - m\delta_{ij}\delta_{\alpha\beta}\right)\Psi_{j\beta} = \overline{\Psi}_{i\alpha}\big(i\not{\partial}_{\alpha\beta} - m\delta_{\alpha\beta}\big)\delta_{ij}\Psi_{j\beta} = \overline{\Psi}_i(i\not{\partial} - m)\delta_{ij}\Psi_j.$$

In the last step, we dropped the Dirac indices again, writing the Dirac matrices and vectors as matrices and vectors instead of their components again.

---

[1] In $d = 4$, γ matrices have dimensionality 4. For general dimensions d, their dimensionality is different. see section 13.2.

According to our derivation in (>15.5.2) and (>15.5.3), the Greens function (times i) of the (free) Dirac operator automatically is the fermion propagator. The "Yang-Mills Dirac operator" differs from the usual Dirac operator only by a unit matrix in symmetry group space, or, in index notation, by a factor δ_{ij}. Thus, are now looking for a Greens function $-i\widetilde{D}_F(z)$, obeying

$$(i\partial\!\!\!/ - m)\left(-i\widetilde{D}_F(z)\right) = \delta(z),$$

or, writing the symmetry group indices,

$$(i\partial\!\!\!/ - m)\delta_{ij}\left(-i\widetilde{D}_{F,jk}(z)\right) = \delta_{ik}\delta(z).$$

It is easy to see, that

$$\widetilde{D}_{F,jk}(z) = \int d^4\bar{p}\frac{i\delta_{jk}}{p\!\!\!/ - m + i\epsilon}e^{-ip\cdot z}$$

solves this equation. Thus,

$$\widetilde{D}_{F,ij}(x-y) = \langle\Omega|\mathcal{T}\,\Psi_i(x)\,\overline{\Psi}_j(y)|\Omega\rangle = \int d^4\bar{p}\frac{i\delta_{ij}}{p\!\!\!/ - m + i\epsilon}e^{-ip\cdot(x-y)},$$

where $\Psi_i = \psi_i$ by definition. Writing also Dirac indices,

$$\widetilde{D}_{F,ij\alpha\beta}(x-y) = \langle\Omega|\mathcal{T}\,\Psi_{i\alpha}(x)\,\overline{\Psi}_{j\beta}(y)|\Omega\rangle = \int d^4\bar{p}\left(\frac{i}{p\!\!\!/ - m + i\epsilon}\right)_{\alpha\beta}\delta_{jk}\,e^{-ip\cdot(x-y)}.$$

18.1.3 Interaction Terms of the Yang-Mills Lagrangian

Consider first the fermion part of the Lagrangian. Plugging in $D_\mu = \partial_\mu + igA_\mu^a t^a$, we find that

$$\overline{\Psi}(iD\!\!\!\!/ - m)\Psi = \cdots + \overline{\Psi}(iigA\!\!\!/^a t^a)\Psi = \cdots - g\overline{\Psi}A_\mu^a\gamma^\mu\Psi t^a,$$

where the dots stand for all the non-interacting terms.

On the other hand, we have also interaction terms in the $F_{\mu\nu}^a F_a^{\mu\nu}$ part of the Lagrangian, since, in contrast to QED, $F_{\mu\nu}^a$ contains a third term, which is quadratic in the gauge fields (see (>18.1.1)):

$$\begin{aligned} F_{\mu\nu}^a F_a^{\mu\nu} &= \left((\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) - gf^{abc}\,A_\mu^b A_\nu^c\right)\left((\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) - gf^{ade}\,A_d^\mu A_e^\nu\right) \\ &= (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) - gf^{ade}\,(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A_d^\mu A_e^\nu \\ &\quad - gf^{abc}\,A_\mu^b A_\nu^c(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) + g^2 f^{abc}f^{ade}\,A_\mu^b A_\nu^c A_d^\mu A_e^\nu. \end{aligned}$$

Here, we have decomposed $F_{\mu\nu}^a F_a^{\mu\nu}$ into four terms. For photons, we have $f^{abc} = 0$ and only the first term survives; it is the kinetic term. In addition, we have three interaction terms, two of which are cubic and one is quartic in the gauge fields.

Let's move on with the cubic terms:

$$\begin{aligned} -gf^{ade}\,(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A_d^\mu A_e^\nu - gf^{abc}\,A_\mu^b A_\nu^c(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) &= -2gf^{abc}\,(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A_b^\mu A_c^\nu \\ = -2g\left(f^{abc}\,(\partial_\mu A_\nu^a)\,A_b^\mu A_c^\nu - f^{abc}\,(\partial_\nu A_\mu^a)\,A_b^\mu A_c^\nu\right)& \\ = -2g\left(f^{abc}\,(\partial_\mu A_\nu^a)\,A_b^\mu A_c^\nu + f^{abc}\,(\partial_\mu A_\nu^a)\,A_c^\nu A_b^\mu\right) &= -4gf^{abc}\,(\partial_\mu A_\nu^a)\,A_b^\mu A_c^\nu. \end{aligned}$$

Here, we used that f^{abc} is totally antisymmetric (see (>2.2.1)). Thus,

$$F_{\mu\nu}^a F_a^{\mu\nu} = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) - 4gf^{abc}\,(\partial_\mu A_\nu^a)\,A_b^\mu A_c^\nu + g^2 f^{abc}f^{ade}\,A_\mu^b A_\nu^c A_d^\mu A_e^\nu.$$

If we now abbreviate the non-interacting terms of the Lagrangian as $\mathcal{L}_0$, we find

$$\mathcal{L}_{\text{YM}} = \mathcal{L}_0 - g\overline{\Psi}A_\mu^a\gamma^\mu\Psi t_a + gf^{abc}\left(\partial_\mu A_\nu^a\right)A_b^\mu A_c^\nu - \frac{1}{4}g^2 f^{abc}f^{ade}\,A_\mu^b A_\nu^c A_d^\mu A_e^\nu$$

(note, that there is a factor of $-1/4$ in front of $F_{\mu\nu}^a F_a^{\mu\nu}$ in the Lagrangian).

18.1.4 The Three Gauge Boson Vertex

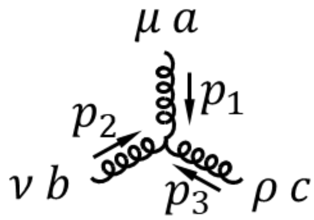
An n-point function is basically given by a matrix element of the form

$$\langle 0|\mathcal{T}\ (\text{external fields})\exp(i\textstyle\int d^4z\,\mathcal{L}_{\text{int}})\,|0\rangle,$$

as we found in section 7.9. In perturbation theory, the exponential will be expanded. There will be terms, containing the three gauge boson vertex term

$$gf^{def}(\partial_\sigma A_\kappa^d)\,A_e^\sigma A_f^\kappa$$

from $\mathcal{L}_{\text{int}}$. Such terms, will always also contain all the external fields and (in higher orders of perturbation theory) might contain additional fields from other vertex terms from $\mathcal{L}_{\text{int}}$. Whatever the term looks like, if it yields a non-vanishing contribution to the n-point function, it must contain three additional gauge boson fields $A_\mu^a, A_\nu^b, A_\rho^c$. They will be contracted with the three gauge boson fields of the vertex term above and – according to Wick's theorem – such contracted pairs become propagators.



Therefore, terms in the expansion of the exponential that contains the three gauge boson vertex looks like

$$\begin{aligned}&\Big\langle\Omega\Big|\left(\cdots A_\mu^a(x_1)\,A_\nu^b(x_2)\,A_\rho^c(x_3)\right)\left(igf^{def}\left(\partial_\sigma A_\kappa^d(z)\right)\,A_e^\sigma(z)\,A_f^\kappa(z)\right)\Big|\Omega\Big\rangle\\ &\qquad= \cdots + \cdots igf^{def}\Big\langle\Omega\Big|A_\mu^a(x_1)\,A_\nu^b(x_2)\,A_\rho^c(x_3)\,\left(\partial_\sigma A_\kappa^d(z)\right)\,A_e^\sigma(z)\,A_f^\kappa(z)\Big|\Omega\Big\rangle,\end{aligned}$$

where the dots stand for other fields which are not connected to this vertex. Applying Wick's theorem to them, they separate from the matrix element written explicitly above. Now, we also apply Wick's theorem to the 6-point function which is left over. How can we form pairs of the six fields? If we contract two of the fields which are evaluated at position z, this corresponds to a diagram where both ends of a gauge boson line are attached to *the same* vertex. This is not, what we want to consider. Thus, we want to contract each of the x_i-fields with one of the z-fields. There are 3! possibilities to do that. According to Wick's theorem, these different possibilities need to be summed up.

Let's start with one of them: Contracting the x_1-field with the first z-field, the x_2-field with the second z-field and the x_3-field with the third z-field yields

$$\begin{aligned}C_1 &:= igf^{def}\Big\langle\Omega\Big|A_\mu^a(x_1)\,\left(\partial_\sigma A_\kappa^d(z)\right)\Big|\Omega\Big\rangle\,\langle\Omega|A_\nu^b(x_2)\,A_e^\sigma(z)|\Omega\rangle\,\langle\Omega|A_\rho^c(x_3)\,A_f^\kappa(z)|\Omega\rangle\\ &= igf^{def}\left(\partial_{z\sigma}\langle\Omega|A_\mu^a(x_1)\,A_\kappa^d(z)|\Omega\rangle\right)\langle\Omega|A_\nu^b(x_2)\,A_e^\sigma(z)|\Omega\rangle\,\langle\Omega|A_\rho^c(x_3)\,A_f^\kappa(z)|\Omega\rangle\\ &= igf^{def}\left(\partial_{z\sigma}\widehat{D}_{F,\mu\kappa}^{ad}(x_1-z)\right)\widehat{D}_{F,\nu e}^{\sigma b}(x_2-z)\,\widehat{D}_{F,\rho f}^{\kappa c}(x_3-z),\end{aligned}$$

where $\partial_{z\sigma}$ is the derivative with respect to the σ component of the position z. In (>18.1.2), we found

$$\widehat{D}_{F,\mu\kappa}^{ad}(x_1-z) = \langle\Omega|A_\mu^a(x_1)\,A_\mu^d(z)|\Omega\rangle = \int d^4\bar{p}_1\,\frac{-i\eta_{\mu\kappa}\delta^{ad}}{p_1^2}\,e^{-ip_1\cdot(x_1-z)}.$$

This propagator describes a particle with momentum p_1 propagating from z (the vertex) to x_1 (the first gauge boson line). Unfortunately, this does not match our picture above, where the momentum $-p_1$ points inwards from x_1 to z. For the sake of consistence with our picture, we turn the integration variable around, $p_1 \to -p_1$:

$$\widehat{D}^{ad}_{F,\mu\kappa}(x_1 - z) = \int d^4\bar{p}_1 \, \frac{-i\eta_{\mu\kappa}\delta^{ad}}{p_1^2} \, e^{ip_1\cdot(x_1-z)} = \int d^4\bar{p}_1 \, \frac{-i\eta_{\mu\kappa}\delta^{ad}}{p_1^2} \, e^{-ip_1\cdot(z-x_1)} = \widehat{D}^{ad}_{F,\mu\kappa}(z - x_1).$$

Now, the propagator describes a particle propagating from x_1 to z with momentum p_1, as desired. We choose this form of the propagators for the rest of this derivation.

Next, consider the derivative of the propagator:

$$\partial_{z\sigma}\widehat{D}^{ad}_{F,\mu\kappa}(z - x_1) = \partial_{z\sigma} \int d^4\bar{p}_1 \, \frac{-i\eta_{\mu\kappa}\delta^{ad}}{p_1^2} \, e^{-ip_1\cdot(z-x_1)} = \int d^4\bar{p} \, \frac{-i\eta_{\mu\kappa}\delta^{ad}}{p^2} \, (ip_{1\sigma}) \, e^{-ip_1\cdot(z-x_1)}$$
$$= \int d^4\bar{p} \, \frac{-i\eta_{\mu\kappa}\delta^{ad}}{p_1^2} \, (-ip_{1\sigma}) \, e^{-ip_1\cdot(z-x_1)}.$$

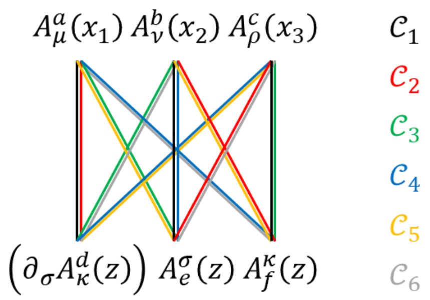
Had we chosen $\widehat{D}^{ad}_{F,\mu\kappa}(x_1 - z)$ instead of $\widehat{D}^{ad}_{F,\mu\kappa}(z - x_1)$, the momentum factor from the derivative would be $+ip_{1\sigma}$, but this momentum p_1 would be *minus* the one in the vertex diagram we drew above.

Next, we switch to momentum space, where

$$\mathcal{C}_1 = igf^{def} \, \frac{-i\eta_{\mu\kappa}\delta^{ad}}{p_1^2} \, (-ip_{1\sigma}) \, \frac{-i\eta^{\sigma}_{\nu}\delta^{b}_{e}}{p_2^2} \, \frac{-i\eta^{\kappa}_{\rho}\delta^{c}_{f}}{p_3^2} = igC^{abc} \, \eta_{\mu\rho}(-ip_{1\nu}) \, \frac{-i}{p_1^2} \, \frac{-i}{p_2^2} \, \frac{-i}{p_3^2}.$$

The fraction $-i/p_i^2$ are part of the propagators. The vertex factor (actually, only its part from the first possible contraction) is simply[1]

$$\mathcal{C}_1 = igf^{def} \left\langle \Omega \middle| A^a_\mu(x_1) \left(\partial_\sigma A^d_\kappa(z)\right) \middle| \Omega \right\rangle \langle \Omega | A^b_\nu(x_2) \, A^\sigma_e(z) | \Omega \rangle \, \langle \Omega | A^c_\rho(x_3) \, A^\kappa_f(z) | \Omega \rangle$$
$$= igf^{abc} \, \eta_{\mu\rho}(-ip_{1\nu}).$$



In the same way, we can evaluate the other five possible contractions. Here, we always drop the factors $-i/p_i^2$ of the propagators and turn implicitly to momentum space. Thus, we write simply $\widehat{D}^{ad}_{F,\mu\kappa}(z - x_1) = \eta_{\mu\kappa}\delta^{ad}$.

$$\mathcal{C}_2 = igf^{def} \left\langle \Omega \middle| A^a_\mu(x_1) \left(\partial_\sigma A^d_\kappa(z)\right) \middle| \Omega \right\rangle \langle \Omega | A^b_\nu(x_2) \, A^\kappa_f(z) | \Omega \rangle \, \langle \Omega | A^c_\rho(x_3) \, A^\sigma_e(z) | \Omega \rangle$$
$$= igf^{def} \left(\partial_{z\sigma}\widehat{D}^{ad}_{F,\mu\kappa}(z - x_1)\right) \widehat{D}^{b\kappa}_{F,\nu f}(z - x_2) \, \widehat{D}^{c\sigma}_{F,\rho e}(z - x_3)$$
$$= igf^{def} \, (-ip_{1\sigma}) \, \eta_{\mu\kappa}\delta^{ad} \, \eta^{\kappa}_{\nu}\delta^{b}_{f} \, \eta^{\sigma}_{\rho}\delta^{c}_{e} = igf^{acb} \, \eta_{\mu\nu}(-ip_{1\rho}),$$

[1] Only the fraction $-i/p_i^2$ is left over here of the propagators. The η's and δ's appear to be absorbed into the vertex factor. Still, we need those η's and δ's in the Feynman rules for the propagators to link the two vertices (or a vertex and a polarization vector, for external particles) to which the gauge boson is attached.

$$\mathcal{C}_3 = igf^{def}\,\langle\Omega|A_\mu^a(x_1)\,A_e^\sigma(z)|\Omega\rangle\,\left\langle\Omega\middle|A_\nu^b(x_2)\,\left(\partial_\sigma A_\kappa^d(z)\right)\middle|\Omega\right\rangle\,\langle\Omega|A_\rho^c(x_3)\,A_f^\kappa(z)|\Omega\rangle$$
$$= igf^{def}\,\eta_\mu^\sigma\delta_e^a\,(-ip_{2\sigma})\eta_{\nu\kappa}\delta^{bd}\,\eta_\rho^\kappa\delta_f^c = igf^{bac}\,\eta_{\nu\rho}(-ip_{2\mu}),$$

$$\mathcal{C}_4 = igf^{def}\,\langle\Omega|A_\mu^a(x_1)\,A_f^\kappa(z)|\Omega\rangle\,\langle\Omega|A_\nu^b(x_2)\,A_e^\sigma(z)|\Omega\rangle\,\left\langle\Omega\middle|A_\rho^c(x_3)\,\left(\partial_\sigma A_\kappa^d(z)\right)\middle|\Omega\right\rangle$$
$$= igf^{def}\,\eta_\mu^\kappa\delta_f^a\,\eta_\nu^\sigma\delta_e^b\,(-ip_{3\sigma})\eta_{\rho\kappa}\delta^{cd} = igf^{cba}\,\eta_{\rho\mu}(-ip_{3\nu}),$$

$$\mathcal{C}_5 = igf^{def}\,\langle\Omega|A_\mu^a(x_1)\,A_e^\sigma(z)|\Omega\rangle\,\langle\Omega|A_\nu^b(x_2)\,A_f^\kappa(z)|\Omega\rangle\,\left\langle\Omega\middle|A_\rho^c(x_3)\,\left(\partial_\sigma A_\kappa^d(z)\right)\middle|\Omega\right\rangle$$
$$= igf^{def}\,\eta_\mu^\sigma\delta_e^a\,\eta_\nu^\kappa\delta_f^b\,(-ip_{3\sigma})\eta_{\rho\kappa}\delta^{cd} = igf^{cab}\,\eta_{\rho\nu}(-ip_{3\mu}),$$

$$\mathcal{C}_6 = igf^{def}\,\langle\Omega|A_\mu^a(x_1)\,A_f^\kappa(z)|\Omega\rangle\,\left\langle\Omega\middle|A_\nu^b(x_2)\,\left(\partial_\sigma A_\kappa^d(z)\right)\middle|\Omega\right\rangle\,\langle\Omega|A_\rho^c(x_3)\,A_e^\sigma(z)|\Omega\rangle$$
$$= igf^{def}\,\eta_\mu^\kappa\delta_f^a\,(-ip_{2\sigma})\eta_{\nu\kappa}\delta^{bd}\,\eta_\rho^\sigma\delta_e^c = igf^{bca}\,\eta_{\nu\mu}(-ip_{2\rho}).$$

Adding up all these contributions and using that f^{abc} is totally antisymmetric (and that $\eta_{\mu\nu}$ is symmetric), we find

$$\begin{aligned}
&\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_4+\mathcal{C}_5+\mathcal{C}_6\\
&\quad= ig\big(f^{abc}\,\eta^{\mu\rho}(-ip_1^\nu)+f^{acb}\,\eta^{\mu\nu}\left(-ip_1^\rho\right)+f^{bac}\,\eta^{\nu\rho}\left(-ip_2^\mu\right)+f^{cba}\,\eta^{\rho\mu}(-ip_3^\nu)\\
&\qquad\qquad +f^{cab}\,\eta^{\rho\nu}\left(-ip_3^\mu\right)+f^{bca}\,\eta^{\nu\mu}\left(-ip_2^\rho\right)\big)\\
&\quad= gf^{abc}\left(\eta^{\mu\rho}p_1^\nu-\eta^{\mu\nu}p_1^\rho-\eta^{\nu\rho}p_2^\mu-\eta^{\rho\mu}p_3^\nu+\eta_{\rho\nu}p_{3\mu}+\eta_{\nu\mu}p_{2\rho}\right)\\
&\quad= gf^{abc}(\eta^{\mu\rho}(p_1-p_3)^\nu+\eta^{\nu\rho}(p_3-p_2)^\mu+\eta^{\mu\nu}(p_2-p_1)^\rho).
\end{aligned}$$

18.1.5 The Four Gauge Boson Vertex

The four gauge boson vertex term reads

$$-\frac{1}{4}g^2 f^{efg}f^{ehi}\,A_\kappa^f A_\eta^g A_h^\kappa A_i^\eta.$$

In the same way as explained lengthily in (>18.1.4), we need to consider all possible contractions with other gauge boson fields. There are now $4!=24$ possible contractions. Exemplary, let's consider only one of them:

$$\begin{aligned}
&-\frac{1}{4}g^2 f^{efg}f^{ehi}\,\langle\Omega|(\cdots A^{a\mu}A^{b\nu}A^{c\rho}A^{d\sigma})(A_\kappa^f A_\eta^g A_h^\kappa A_i^\eta)|\Omega\rangle \overset{\text{only 1}}{\underset{\text{contraction}}{=}}\\
&\quad -\frac{1}{4}g^2 f^{efg}f^{ehi}\,\langle\Omega|A^{a\mu}\,A_\kappa^f|\Omega\rangle\,\langle\Omega|A^{b\nu}\,A_\eta^g|\Omega\rangle\,\langle\Omega|A^{c\rho}\,A_h^\kappa|\Omega\rangle\,\langle\Omega|A^{d\sigma}\,A_i^\eta|\Omega\rangle\\
&\quad= -\frac{1}{4}g^2 f^{efg}f^{ehi}\,\eta_\kappa^\mu\delta^{af}\,\eta_\eta^\nu\delta^{bg}\,\eta^{\rho\kappa}\delta_h^c\,\eta^{\sigma\eta}\delta_i^d = -\frac{1}{4}g^2 f^{eab}f^{ecd}\,\eta^{\rho\mu}\,\eta^{\sigma\nu}.
\end{aligned}$$

In the same way, all other contraction will give similar results. Indeed, sets of four of them are equal, such that the factor $1/4$ is cancelled and only $4!/4=6$ different terms remain:

$$\begin{aligned}
&-ig^2\Big(f^{abe}f^{ecd}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho})+f^{ace}f^{ebd}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho})\\
&\quad+f^{ade}f^{ebc}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma})\Big).
\end{aligned}$$

18.2 The Faddeev-Popov Lagrangian

18.2.1 The Gauge Boson Propagator

Consider a theory with gauge bosons only, that is a Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu}.$$

Due to our discussion in chapter 14, the n-point function of fields can be written as some fraction of functional integrals, the denominator of which contains the expression

$$\int \mathcal{D}A \exp\left(i \int d^4x\, \mathcal{L}[A]\right) = \int \mathcal{D}A\, e^{iS[A]},$$

where $\mathcal{D}A \coloneqq \Pi_{a,\mu}\, \mathcal{D}A_\mu^a$. $S[A]$ is the action. Since we know from section 3.7, that $\mathcal{L}$ is invariant under the (infinitesimal) gauge transformation[1]

$$A_\mu^a \to (A^\alpha)_\mu^a \coloneqq A_\mu^a - \frac{1}{g}\left(\partial_\mu \alpha^a\right) + f^{abc} A_\mu^b \alpha^c = A_\mu^a - \frac{1}{g}\left(D_\mu \alpha\right)^a,$$

the functional integral is badly defined, as explained in more detail in (>15.3.1). Imposing arbitrary gauge conditions $G^a(A) = 0\ \forall a$ will constrain the fields A^a covered by the functional integral to one certain set of gauge equivalent fields. As in (>15.3.2), we impose this constraint by inserting a 1 in form of[2]

$$1 = \int \mathcal{D}\alpha\, \delta\big(G^a(A^\alpha)\big) \left|\det \frac{\delta G^c(A^\alpha)}{\delta \alpha^d}\right|,$$

into the functional integral:

$$\int \mathcal{D}A\, e^{iS[A]} = \int \mathcal{D}A\, \mathcal{D}\alpha\, e^{iS[A^\alpha]}\, \delta\big(G^a(A^\alpha)\big) \left|\det \frac{\delta G^c(A^\alpha)}{\delta \alpha^d}\right|.$$

Since the Lagrangian is gauge invariant, we also replaced $S[A] \to S[A^\alpha]$ in this step.

Next, we should choose a gauge condition. Let's choose the generalized Lorentz condition

$$G^c(A) = \partial^\mu A_\mu^c(x) - \omega^c(x).$$

$$\Longrightarrow \qquad \frac{\delta G^c(A^\alpha)}{\delta \alpha^d} = \frac{\delta}{\delta \alpha^d}\left(\partial^\mu \left(A_\mu^c - \frac{1}{g}\left(D_\mu \alpha\right)^c\right) - \omega^c\right) = -\frac{1}{g}\frac{\delta}{\delta \alpha^d}\left(\partial^\mu D_\mu^{cb} \alpha_b\right) = -\frac{1}{g}\partial^\mu D_\mu^{cd}.$$

We can now shift the integration variable $A \to A + g^{-1}(\partial\alpha) - f_{bc} A^b \alpha^c$ (actually, this is a shift plus a unitary rotation, which also does not change the measure). This will turn $A^\alpha \to A$ and will leave D_μ unchanged (to leading order in the infinitesimal α):

$$\int \mathcal{D}A\, e^{iS[A]} = \int \mathcal{D}A\, \mathcal{D}\alpha\, e^{iS[A]}\, \delta\big(G^a(A)\big) \left|\det \frac{1}{g}\partial^\mu D_\mu^{cd}\right|.$$

The minus sign inside the functional determinant is irrelevant because only the absolute value of the determinant is taken. The only important difference to the photon field quantization from (>15.3.2) is that the determinant now depends on A, whereas it was constant for the photon field.

Since the equation above holds for *arbitrary* functions ω^a it must also hold for a (properly normalized) linear combination of different functions ω_i^a with coefficients $C_i(\omega_i^a)$. For any a, we can introduce such a linear combination. Instead of the sum over i, we can perform an integral over ω^a and choose the normalizations factors $C(\omega^a)$ to be a Gaussian function together with some normalization factor N. Finally, we use the δ-function to get rid of ω:

---

[1] We used here, that in the adjoint representation $(t^a)_{bc} = -if^{abc}$ from section 2.2,

$$\left(D_\mu \alpha\right)^a = \left(\partial_\mu \alpha + igA_\mu^b t^b \alpha\right)^a = \partial_\mu \alpha^a + igA_\mu^b (t^b)_{ac} \alpha^c = \partial_\mu \alpha^a + gA_\mu^b f^{bac} \alpha^c = \partial_\mu \alpha^a - gA_\mu^b f^{abc} \alpha^c$$

[2] The notation is a little bit sloppy at this point: The δ-function $\delta\big(G^a(A)\big)$ is actually $N^2 - 1$ δ-functions:

$$\delta\big(G^a(A)\big) \coloneqq \Pi_{a=1}^{N^2-1}\, \delta\big(G^a(A)\big).$$

One for each value of a. And the determinant is in fact taken of a *matrix* $\delta G(A^\alpha)/\delta\alpha$, although we denote this matrix in terms of its elements with indices c and d.

$$\begin{aligned}
\int \mathcal{D}A\, e^{iS[A]} &= \int \mathcal{D}A\, \mathcal{D}\alpha\, e^{iS[A]}\, \delta\left(\partial^\mu A^a_\mu(x) - \omega^a(x)\right) \left|\det \frac{1}{g} \partial^\mu D^{cd}_\mu\right| \\
&= \sum_{i\ldots k} C_i(\omega^1_i) \cdots C_k(\omega^{N^2-1}_k) \int \mathcal{D}A\, \mathcal{D}\alpha\, e^{iS[A]}\, \delta\left(\partial^\mu A^a_\mu(x) - \omega^a_i(x)\right) \left|\det \frac{1}{g} \partial^\mu D^{cd}_\mu\right| \\
&= N(\xi) \int \mathcal{D}\omega \left(\prod_{b=1}^{N^2-1} \exp\left(-i \int d^4x \frac{(\omega^b)^2}{2\xi}\right)\right) \\
&\qquad\qquad \int \mathcal{D}A\, \mathcal{D}\alpha\, e^{iS[A]}\, \delta\left(\partial^\mu A^a_\mu(x) - \omega^a(x)\right) \left|\det \frac{1}{g} \partial^\mu D^{ad}_\mu\right| \\
&= N(\xi) \int \mathcal{D}\omega \exp\left(-i \int d^4x \frac{\omega^b \omega_b}{2\xi}\right) \int \mathcal{D}A\, \mathcal{D}\alpha\, e^{iS[A]}\, \delta\left(\partial^\mu A^a_\mu(x) - \omega^a(x)\right) \left|\det \frac{1}{g} \partial^\mu D^{ad}_\mu\right| \\
&= N(\xi) \int \mathcal{D}A\, \mathcal{D}\alpha\, \exp\left(-i \int d^4x \frac{1}{2\xi} \left(\partial^\mu A^b_\mu\right)\left(\partial^\nu A_{b\nu}\right)\right) e^{iS[A]} \left|\det \frac{1}{g} \partial^\mu D^{ad}_\mu\right|.
\end{aligned}$$

Effectively, we have added a new kinetic term

$$-\frac{1}{2\xi}\left(\partial^\mu A^b_\mu\right)\left(\partial^\mu A_{b\mu}\right)$$

to the Lagrangian. It now reads

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)\left(\partial^\nu A_{a\nu}\right) \\
&= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)\left(\partial^\nu A_{a\nu}\right) + \mathcal{O}(A^3) \\
&= -\frac{1}{2}\left(\left(\partial_\mu A^a_\nu\right)\left(\partial^\mu A^\nu_a\right) - \left(\partial_\nu A^a_\mu\right)\left(\partial^\mu A^\nu_a\right)\right) - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)\left(\partial_\nu A^\nu_a\right) + \mathcal{O}(A^3) \\
&= -\frac{1}{2}\left(-A^a_\nu \Box A^\nu_a + A^a_\mu \partial_\nu \partial^\mu A^\nu_a\right) + \frac{1}{2\xi} A^a_\mu \partial^\mu \partial_\nu A^\nu_a + \mathcal{O}(A^3) \\
&= -\frac{1}{2} A^a_\mu \delta^b_a \left(-\eta^{\mu\nu}\Box + (1 - \xi^{-1})\partial^\nu \partial^\mu - i\epsilon\right) A_{b\nu} + \mathcal{O}(A^3),
\end{aligned}$$

where we used integration by parts, since Lagrangians appear under integrals. The sudden appearance of the $i\epsilon$ is explained in (>15.2.4). This is exactly the operator as finally found in (>15.3.2), except for the trivial new factor δ^b_a. In Fourier space, it reads

$$\left(\delta^{ab}\left(\eta_{\mu\nu} k^2 - (1 - \xi^{-1}) k_\mu k_\nu - i\epsilon\right)\right) i \widehat{D}^{\nu\sigma}_{F,bc} = \delta^\sigma_\mu \delta^a_c,$$

which is solved by (see also (>15.3.2))

$$\widehat{D}^{\nu\sigma}_{F,bc}(k) = \frac{-i\delta_{bc}}{k^2 + i\epsilon}\left(\eta^{\nu\sigma} - (1 - \xi)\frac{k^\nu k^\sigma}{k^2}\right).$$

18.2.2 Faddeev-Popov Ghosts

The derivation of the gauge boson propagator from (>18.2.1) was almost the same as of the photon propagator from (>15.3.2). However, there is one important difference: The functional determinant now depends on A. Thus, is cannot be treated as just another constant like N that is cancelled, when we are interested in n-point functions (which are a fraction of functional integrals, as we saw in (>15.1.2). This cancellation is no longer possible in the non-Abelian case.

The determinant now reads (see footnote on page 187)

$$\left|\det \frac{1}{g} \partial^\mu D^{ac}_\mu\right|, \qquad D^{ac}_\mu = \delta^{ac}\partial_\mu + igA^b_\mu (t_b)^{ac} = \delta^{ac}\partial_\mu + gA^b_\mu f^{bac} = \delta^{ac}\partial_\mu - gA^b_\mu f^{abc}.$$

Using the analogy $\int (\Pi_i d\theta^*_i d\theta_i) e^{-\theta^*_i A_{ij} \theta_j} = \det A$, as we already did in (>15.5.2), we can write

$$\left|\det\frac{1}{g}\,\partial^\mu D_\mu\right| = \left|\det\frac{i}{g}\,\partial^\mu D_\mu\right| = \int \mathcal{D}\vartheta\,\mathcal{D}\bar{\vartheta}\,\exp\left(i\int d^4x\,\bar{\vartheta}\left(-\partial^\mu D_\mu\right)\vartheta\right).$$

The factor $1/g$ is absorbed into the Graßmann fields ϑ. We can write them as another effective addition to the Lagrangian. Thereby, ϑ can give rise to new particle excitations. They are called *ghosts*. Their contribution to the Lagrangian is

$$\mathcal{L}_{\text{ghosts}} = \bar{\vartheta}\left(-\partial^\mu D_\mu\right)\vartheta = \bar{\vartheta}^a\left(-\partial^\mu\left(\delta_{ac}\partial_\mu - gA_\mu^b C_{abc}\right)\right)\vartheta^c = -\bar{\vartheta}^a\left(\delta_{ac}\Box - gf^{abc}\partial^\mu A_\mu^b\right)\vartheta^c.$$

Note, that the derivative ∂^μ in the second term not only acts on A_μ^b, but also on ϑ^c. The kinetic operator in the Lagrangian, $\delta^{ab}\Box$, is equivalent to $\delta^{ab}k^2$ in Fourier space. The kinetic operator of the gauge boson at the end of (>18.2.1) reads $-\delta_{ab}\eta^{\mu\nu}\Box + \cdots$. Obviously, the ghost propagator in Fourier space reads

$$\breve{D}_{F,ab}(k) \coloneqq \frac{i\delta_{ab}}{k^2}.$$

The second term of $\mathcal{L}_{\text{ghosts}}$ describes an interaction between two ghosts and a gauge boson. This interaction part of the ghost Lagrangian can be written as

$$\mathcal{L}_{\text{int}} = gf^{def}\,\bar{\vartheta}^d\partial^\mu A_\mu^e\vartheta^f = gf^{def}\,\bar{\vartheta}^d\left(\vartheta^f\,\partial^\mu A_\mu^e + A_\mu^e\,\partial^\mu\vartheta^f\right).$$

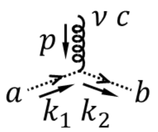
We can find the corresponding Feynman rule for this vertex in the same way as in (>18.1.4):

$$\begin{aligned}&\langle 0|\mathcal{T}\ (\text{external fields})\ \exp(i\textstyle\int d^4z\,\mathcal{L}_{\text{int}})\,|0\rangle\\ &\quad= \left\langle\Omega\right|\left(\cdots\bar{\vartheta}_a(x_1)\,\vartheta_b(x_2)\,A_{c\nu}(x_3)\right)\left(igf^{def}\,\bar{\vartheta}^d(z)\,\left(\vartheta^f(z)\,\partial^\mu A_\mu^e(z) + A_\mu^e(z)\,\partial^\mu\vartheta^f(z)\right)\right)\left|\Omega\right\rangle.\end{aligned}$$

There is only one possible contraction, since a ghost propagator needs to be composed out of a η field and a $\bar{\vartheta}$ field. Note, that the order is important; first ϑ, then $\bar{\vartheta}$: $\breve{D}_{F,ab}(x-y) = \left\langle\Omega\right|\mathcal{T}\,\vartheta_a(x)\,\bar{\vartheta}_b(y)\left|\Omega\right\rangle$. Thus, whereas for the gauge bosons $\widehat{D}_F(x-z) = \widehat{D}_F(z-x)$, this is not true for the ghosts. Contracting the three pairs of fields and form propagators out of them yields

$$igf^{def}\,\breve{D}_{F,b}^d(x_2 - z)\,\left(-\breve{D}_{F,a}^f(z-x_1)\,\partial_z^\mu\widehat{D}_{F,c\mu\nu}^e(z-x_3) - \widehat{D}_{F,c\mu\nu}^e(z-x_3)\,\partial_z^\mu\breve{D}_{F,a}^f(z-x_1)\right),$$

where the minus signs in the brackets come from commuting $\bar{\vartheta}_a$ with ϑ^f. The derivatives ∂_z^μ yield a factor of $-ip^\mu$, where p^μ is the momentum of the propagator. To assign certain momenta to certain particles, let us denote the momenta in the following way:



The dotted lines are the ghosts. In the first term in the brackets, the derivative acts on the gauge boson propagator, describing a propagation from x_3 to z (to the vertex), which coincides with the direction of the momentum p, thus we get a factor $-ip^\mu$ (and not ip^μ). In the second term, the derivative gives a factor $-ik_1^\mu$. As explained in more detail in in (>18.1.4), we then turn to momentum space and keep only the δ's and η's from the propagators:

$$\begin{aligned}&igf^{def}\,\delta_b^d\,\left(-\delta_a^f\,(-ip^\mu)\,\delta_c^e\eta_{\mu\nu} - \delta_c^e\eta_{\mu\nu}\,\left(-ik_1^\mu\right)\delta_a^f\right) = i^2gf^{def}\,\delta_b^d\,\delta_a^f\delta_c^e\eta_{\mu\nu}\left(p^\mu + k_1^\mu\right)\\ &\quad= i^2gf^{bca}\,(p+k_1)_\nu = -gf^{abc}\,k_{2\nu}.\end{aligned}$$

In the last step, we used momentum conservation: $p + k_1 = k_2$.

18.2.3 Note: No Ghosts in Axial Gauge (no reference in summary document)

In the derivations above, we used the generalized Lorentz gauge

$$G^c(A) = \partial^\mu A^a_\mu(x) - \omega^c(x).$$

Let us, instead, consider the axial gauge

$$G^c(A) = n^\mu A^c_\mu(x) - \omega^c(x),$$

for some vector n^μ with $n^2 = 1$. Using this gauge, we find

$$\frac{\partial G^c(A^\alpha)}{\partial \alpha^d} = \frac{\partial}{\partial \alpha^d} n^\mu \left(A^c_\mu - \frac{1}{g}\partial_\mu \alpha^c + f^{cab} A^a_\mu \alpha^b\right) = \frac{\partial}{\partial \alpha^d}\left(n^\mu A^c_\mu - \frac{1}{g} n^\mu \partial_\mu \alpha^c + f^{cab} n^\mu A^a_\mu \alpha^b\right)$$
$$= -\frac{1}{g} n^\mu \partial_\mu \delta^{cd} + f^{cad} n^\mu A^a_\mu = -\frac{1}{g} n^\mu D^{cd}_\mu$$

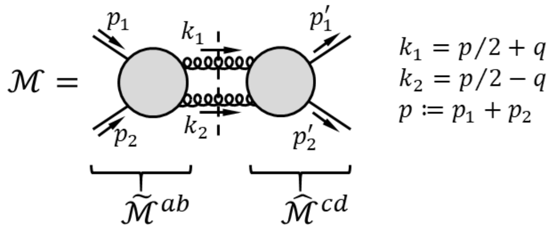
$$D^{bc}_\mu = \delta^{bc}\partial_\mu + g f^{abc} A^a_\mu$$

$$\frac{\delta G^c(A^\alpha)}{\delta \alpha^d} = \frac{\delta}{\delta \alpha^d}\left(\partial^\mu \left(A^c_\mu - \frac{1}{g}\left(D_\mu \alpha\right)^c\right) - \omega^c\right) = -\frac{1}{g}\frac{\delta}{\delta \alpha^d}\left(\partial^\mu D^{cb}_\mu \alpha_b\right) = -\frac{1}{g}\partial^\mu D^{cd}_\mu$$
$$= -\frac{1}{g}\partial^\mu\left(\delta^{cd}\partial_\mu + g f^{acd} A^a_\mu\right)$$

18.3 Ghosts to Fix the Optical Theorem

18.3.1 Cutting the Diagram

We want to cut the following diagram along the dashed line:



This cut produces an amplitude $\widetilde{\mathcal{M}}^{ab}$, describing fermion-antifermion annihilation into two gauge bosons and an amplitude $\widehat{\mathcal{M}}^{cd}$ describing two gauge bosons producing a fermion-antifermion pair. In the diagram $\mathcal{M}$ (without the cut), those two amplitudes are connected by propagators of the two intermediate gauge bosons. That is, the amplitude $\mathcal{M}$ reads

$$i\mathcal{M} = \frac{1}{2}\int d^4\bar{q}\,\left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\right)\frac{-i\eta^{\mu\rho}\delta_{ac}}{k_1^2 + i\epsilon}\frac{-i\eta^{\nu\sigma}\delta_{bd}}{k_2^2 + i\epsilon}\left(i\widehat{\mathcal{M}}^{cd}_{\rho\sigma}\right)$$
$$= \frac{1}{2}\int d^4\bar{q}\,\left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\right)\frac{-i\eta^{\mu\rho}\delta_{ac}}{(p/2+q)^2 + i\epsilon}\frac{-i\eta^{\nu\sigma}\delta_{bd}}{(p/2-q)^2 + i\epsilon}\left(i\widehat{\mathcal{M}}^{cd}_{\rho\sigma}\right).$$

There are few things to clarify about this expression. First, note that the notation of the momenta is exactly the same as in section 11.3. Second, the amplitudes $\widetilde{\mathcal{M}}_{\mu\nu}$ and $\widehat{\mathcal{M}}_{\rho\sigma}$ are here used in the form *without* the adjacent gauge bosons. That is to say, just as our usual notation when dealing with the Ward identity, we write

$$i\widetilde{\mathcal{M}}^{ab} = i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\,\varepsilon^\mu_{k_1}\varepsilon^\nu_{k_2}, \qquad i\widehat{\mathcal{M}}^{cd} = i\widehat{\mathcal{M}}^{cd}_{\mu\nu}\,\varepsilon^\mu_{k_1}\varepsilon^\mu_{k_2}.$$

Finally, the global factor $1/2$ is a symmetry factor.

In (>11.3.5), we derived that we can write the equation above also in terms of two integrals over k_1 and k_2 instead of over q:

$$i\mathcal{M} = \int d^4\bar{k}_1\, d^4\bar{k}_2\, (2\pi)^4\delta(k_1 + k_2 - p)\, \left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\right)\frac{-i\eta^{\mu\rho}\delta_{ac}}{k_1^2 + i\epsilon}\frac{-i\eta^{\nu\sigma}\delta_{bd}}{k_2^2 + i\epsilon}\left(i\widehat{\mathcal{M}}^{cd}_{\rho\sigma}\right).$$

Now it is time to cut the diagram, which means, we replace the propagators by δ-functions as in section 11.3 and 11.4:

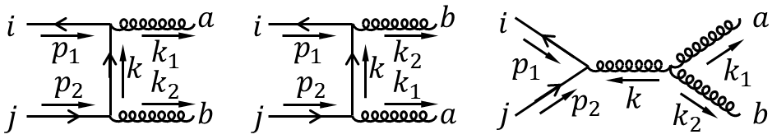
$$\frac{1}{k_i^2 + i\epsilon} \to -2\pi i\, \delta\left(k_i^2\right).$$

Doing this replacement in the amplitude gives us the imaginary part of the amplitude. Specifically, we also need to replace $i\mathcal{M} \to 2\,\mathrm{Im}\,\mathcal{M}$ to keep a valid equation. Then, as we saw in (>11.3.6), we can use the δ-functions to turn $d^4\bar{k}_i$ into $d\tilde{k}_i$:

$$\begin{aligned}
&2\,\mathrm{Im}\,\mathcal{M}\\
&\quad= \frac{1}{2}\int d^4\bar{k}_1\, d^4\bar{k}_2\, (2\pi)^4\delta(k_1 + k_2 - p)\\
&\qquad\qquad \left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\right)\left(-i\eta^{\mu\rho}\delta_{ac}(-2\pi i)\delta(k_1^2)\right)\left(-i\eta^{\nu\sigma}\delta_{bd}(-2\pi i)\delta(k_2^2)\right)\left(i\widehat{\mathcal{M}}^{cd}_{\rho\sigma}\right)\\
&\quad= \frac{1}{2}\int \underbrace{d\tilde{k}_1\, d\tilde{k}_2\, (2\pi)^4\delta(k_1 + k_2 - p)}_{=:d\phi}\, \left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\right)\left(-i\eta^{\mu\rho}\delta_{ac}(-i)\right)\left(-i\eta^{\nu\sigma}\delta_{bd}(-i)\right)\left(i\widehat{\mathcal{M}}^{cd}_{\rho\sigma}\right)\\
&\quad= \frac{1}{2}\int d\phi\, \left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\right)\eta^{\mu\rho}\eta^{\nu\sigma}\delta_{ac}\delta_{bd}\left(i\widehat{\mathcal{M}}^{cd}_{\rho\sigma}\right).
\end{aligned}$$

18.3.2 Amplitude of Fermion Annihilation to Gauge Boson Production

To order g^2, the amplitude receives contributions from three Feynman diagrams: $\widetilde{\mathcal{M}}^{ab} = \sum_{i=1}^{3}\widetilde{\mathcal{M}}^{ab}_i$.



$\widetilde{\mathcal{M}}^{ab}_1$ $\quad$ $\widetilde{\mathcal{M}}^{ab}_2$ $\quad$ $\widetilde{\mathcal{M}}^{ab}_3$

THE FIRST TWO DIAGRAMS:

Using Feynman rules, the first two diagrams yield[1]

$$i\widetilde{\mathcal{M}}^{ab}_1 = \bar{v}_{p_1}\;\left(-ig\gamma^\mu t^a\right)\;\frac{i}{\not{k} - m}\;\left(-ig\gamma^\nu t^b\right)\;u_{p_2}\;\;\varepsilon_{\mu k_1}\varepsilon_{\nu k_2},\qquad k = p_2 - k_2,$$

$$i\widetilde{\mathcal{M}}^{ab}_2 = \bar{v}_{p_1}\;\left(-ig\gamma^\mu t^b\right)\;\frac{i}{\not{k} - m}\;\left(-ig\gamma^\nu t^a\right)\;u_{p_2}\;\;\varepsilon_{\mu k_2}\varepsilon_{\nu k_1},\qquad k = k_2 - p_1.$$

Adding them up yields

$$\begin{aligned}
&i\widetilde{\mathcal{M}}^{ab}_{12} \coloneqq i\widetilde{\mathcal{M}}^{ab}_1 + i\widetilde{\mathcal{M}}^{ab}_2\\
&\qquad = (-ig)^2\;\;\bar{v}_{p_1}\;\;\left(\not{\varepsilon}_{k_1} t^a\;\frac{i}{\not{p}_2 - \not{k}_2 - m}\;\not{\varepsilon}_{k_2} t^b\;\;+\;\;\not{\varepsilon}_{k_2} t^b\;\frac{i}{\not{k}_2 - \not{p}_1 - m}\;\not{\varepsilon}_{k_1} t^a\right)\;u_{p_2}.
\end{aligned}$$

THE THIRD DIAGRAM:

[1] We do not write the indices of the symmetry space. Thus, the Kronecker delta of the fermion propagator is a unit matrix which is not written explicitly. Similarly, the matrix components t^a_{ij} appear without indices as matrices t^a.

Here, the three gauge boson vertex comes in. Since the momenta are outwards in this case, we need a total minus sign in the vertex rule of the three gauge boson vertex (where momenta were defined inwards). Note, that $-k = k_1 + k_2$.

$$\begin{aligned} i\widetilde{\mathcal{M}}_3^{ab} &= \bar{v}_{p_1} \ \left(-ig\gamma^\sigma t^d\right) \ \frac{-i\eta_{\sigma\rho}\delta_{dc}}{k^2} \\ &\qquad \cdot \left(-gf^{abc}(\eta^{\mu\nu}(k_2 - k_1)^\rho + \eta^{\mu\rho}(k_1 - k)^\nu + \eta^{\nu\rho}(k - k_2)^\mu)\right) \ u_{p_2} \ \varepsilon_{\mu k_1}\varepsilon_{\nu k_2} \\ &= (ig^2) \ \bar{v}_{p_1} \ \left(\gamma_\rho t_c\right) \ \frac{-i}{k^2} \\ &\qquad \cdot f^{abc}\left(\left(\varepsilon_{k_1} \cdot \varepsilon_{k_2}\right)(k_2 - k_1)^\rho + \varepsilon_{k_1}^\rho\left(\varepsilon_{k_2} \cdot (k_1 - k)\right) + \varepsilon_{k_2}^\rho\left(\varepsilon_{k_1} \cdot (k - k_2)\right)\right) \ u_{p_2}. \end{aligned}$$

AMPLITUDE OF GAUGE BOSON ANNIHILATION TO FERMION PRODUCTION

$\widehat{\mathcal{M}}^{cd}$ is simply the inverse process to $\widetilde{\mathcal{M}}^{ab}$. One can convince oneself that we can get $\widehat{\mathcal{M}}^{cd}$ from $\widetilde{\mathcal{M}}^{ab}$ simply by the replacements

$$k_{1,2} \to -k_{1,2}, \qquad \bar{v}_{p_1} \to \bar{u}_{p_2'}, \qquad u_{p_2} \to v_{p_1'},$$

where we choose p_1' to be the outgoing momentum of the antifermion and p_2' of the fermion.

18.3.3 Choice of Polarizations

Let $k^\mu = \left(k^0, \vec{k}\right)$ be the momentum of the gauge boson. Then, we can choose the two physical polarizations $\varepsilon_{\lambda k}^\mu$ for $\lambda = 1, 2$ to be purely spatial vectors orthogonal to $\vec{k}$ and to each other:

$$\varepsilon_{\lambda k}^\mu = \begin{pmatrix} 0 \\ \vec{k}_{\perp\lambda} \end{pmatrix}, \qquad \text{where} \qquad \vec{k}_{\perp\lambda} \cdot \vec{k} = 0, \qquad \vec{k}_{\perp\lambda} \cdot \vec{k}_{\perp\lambda'} = \delta_{\lambda\lambda'}.$$

Thus, also $\varepsilon_{\lambda k} \cdot k = 0$.

For the present purpose, it is most convenient to use

$$\varepsilon_{\pm k}^\mu = \frac{1}{\sqrt{2}|\vec{k}|}\begin{pmatrix} k^0 \\ \pm\vec{k} \end{pmatrix}$$

for the unphysical polarizations. Then we have the following identities[1]

$$\varepsilon_{\lambda k} \cdot \varepsilon_{\lambda' k} = -\delta_{\lambda\lambda'}, \qquad \varepsilon_{\pm k} \cdot \varepsilon_{\lambda k} = 0, \qquad \varepsilon_{\pm k} \cdot \varepsilon_{\pm' k} = 1 - \delta_{\pm\pm'}.$$

It is easy to show explicitly, that they also obey the completeness relation

$$\eta^{\mu\nu} = \varepsilon_{-k}^\mu \varepsilon_{+k}^\nu + \varepsilon_{+k}^\mu \varepsilon_{-k}^\nu - \sum_{\lambda=1,2} \varepsilon_{\lambda k}^\mu \varepsilon_{\lambda k}^\nu.$$

18.3.4 Plugging in the Expansion in Polarizations

Plugging in this expansion for the η's from (>18.3.3), the following expression which appears inside $2\,\mathrm{Im}\,\mathcal{M}$ reads

[1] Derivation of the last identity: Since k is a gauge boson momentum, we have $0 = k^2 = (k^0)^2 - \vec{k}^2$. Thus,

$$\varepsilon_{\pm k} \cdot \varepsilon_{\pm' k} = \frac{1}{2\vec{k}^2}\left((k^0)^2 - (\pm\pm')\vec{k}^2\right).$$

If the signs $\pm$ and $\pm'$ are equal, the bracket vanishes. Otherwise, the brackets give $(k^0)^2 + \vec{k}^2 = 2\vec{k}^2$. Thus, $\varepsilon_{\pm k} \cdot \varepsilon_{\pm' k}$ vanishes for equal signs and gives 1 for unequal signs.

$$\frac{1}{2}\left(i\widetilde{\mathcal{M}}_{\mu\nu}^{ab}\right)\eta^{\mu\rho}\eta^{\nu\sigma}\delta_{ac}\delta_{bd}\left(i\widehat{\mathcal{M}}_{\rho\sigma}^{cd}\right)$$
$$= \frac{1}{2}\left(i\widetilde{\mathcal{M}}_{\mu\nu}^{ab}\right)\left(\varepsilon_{-k_1}^{\mu}\varepsilon_{+k_1}^{\rho} + \varepsilon_{+k_1}^{\mu}\varepsilon_{-k_1}^{\rho} - \sum_{\lambda=1,2}\varepsilon_{\lambda k_1}^{\mu}\varepsilon_{\lambda k_1}^{\rho}\right)$$
$$\left(\varepsilon_{-k_2}^{\nu}\varepsilon_{+k_2}^{\sigma} + \varepsilon_{+k_2}^{\nu}\varepsilon_{-k_2}^{\sigma} - \sum_{\lambda'=1,2}\varepsilon_{\lambda' k_2}^{\nu}\varepsilon_{\lambda' k_2}^{\sigma}\right)\delta_{ac}\delta_{bd}\left(i\widehat{\mathcal{M}}_{\rho\sigma}^{cd}\right).$$

When we multiply out those two large brackets, there will be sixteen terms (the sums over λ and λ' contains two terms each).

18.3.5 Cancellation of Terms without the Need of Ghosts

Some of the terms from (>18.3.4) that involve unphysical polarizations cancel right away, also without the ghosts (however not all of them; the ghosts will be needed in the end to cancel them all). Let's consider those terms now.

To show that these terms vanish, we need the following statement: At the end of the present section (see below) we are going to show, that the expression

$$i\widetilde{\mathcal{M}}_{\mu\nu}^{ab}\varepsilon_{k_1}^{\mu}k_2^{\nu}$$

vanishes, *if* the first polarization vector obeys $\varepsilon_{k_1}\cdot k_1 = 0$ (we encapsulate this proof into a subsection at the end of the present section (>18.3.5) for a better overview).

When we multiply out the large brackets in (>18.3.4), in each of the sixteen terms the amplitude $\widetilde{\mathcal{M}}_{\mu\nu}^{ab}$ will be contracted with one polarization factor of the first brackets (carrying a μ and k_1) and one of the second bracket (carrying a ν and k_2).

For example, there will be a term that contains the expression

$$i\widetilde{\mathcal{M}}_{\mu\nu}^{ab}\varepsilon_{\lambda k_1}^{\mu}\varepsilon_{+k_2}^{\nu}.$$

This term vanishes, since by the definition of our polarization vectors from (>18.3.3), $\varepsilon_{+k_2}^{\mu} = k_2^{\mu}/\sqrt{2}|\vec{k}_2| \sim k_2^{\mu}$. Also, the physical polarization $\varepsilon_{\lambda k_1}^{\mu}$ fulfills $\varepsilon_{\lambda k_1}\cdot k_1 = 0$. Therefore, using the statement with the pending proof from above, this term vanishes.

Also terms, where both polarizations are $+$-polarizatons, vanish, since also $\varepsilon_{+k_1}^{\mu}$ fulfils the necessary condition $\varepsilon_{+k_1}^{\mu}\cdot k_1 \sim k_1^2 = 0$:

$$i\widetilde{\mathcal{M}}_{\mu\nu}^{ab}\varepsilon_{+k_1}^{\mu}\varepsilon_{+k_2}^{\nu} \sim i\widetilde{\mathcal{M}}_{\mu\nu}^{ab}\varepsilon_{+k_1}^{\mu}k_2^{\nu} = 0.$$

It can be shown, that also all other terms vanish, except for terms with $+$ *and* $-$ polarization as well as terms with only physical polarizations. Thus, the only terms that remain are

$$\frac{1}{2}\left(i\widetilde{\mathcal{M}}_{\mu\nu}^{ab}\right)\eta^{\mu\rho}\eta^{\nu\sigma}\delta_{ac}\delta_{bd}\left(i\widehat{\mathcal{M}}_{\rho\sigma}^{cd}\right)$$
$$= \frac{1}{2}\left(i\widetilde{\mathcal{M}}_{\mu\nu}^{ab}\right)\left(\varepsilon_{-k_1}^{\mu}\varepsilon_{+k_1}^{\rho}\varepsilon_{+k_2}^{\nu}\varepsilon_{-k_2}^{\sigma} + \varepsilon_{+k_1}^{\mu}\varepsilon_{-k_1}^{\rho}\varepsilon_{-k_2}^{\nu}\varepsilon_{+k_2}^{\sigma} + \varepsilon_{\lambda k_1}^{\mu}\varepsilon_{\lambda k_1}^{\rho}\varepsilon_{\lambda' k_2}^{\nu}\varepsilon_{\lambda' k_2}^{\sigma}\right)\delta_{ac}\delta_{bd}\left(i\widehat{\mathcal{M}}_{\rho\sigma}^{cd}\right),$$

where a sum over λ and λ' is implied. Note, that the third terms is the one with only physical polarizations, which is needed to satisfy the optical theorem. The other two terms will be cancelled only by the ghosts.

PROOF OF THE STATEMENT GIVEN IN THE BEGINNING:

We now want to proof that

$$i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{k_1}k_2^{\nu} = 0 \qquad \text{if} \qquad \varepsilon_{k_1}\cdot k_1 = 0.$$

For this purpose, we go back to the explicit expression for the amplitude $\widetilde{\mathcal{M}}^{ab} = \widetilde{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{k_1}\varepsilon^{\nu}_{k_2}$ from (>18.3.2) and replace $\varepsilon_{k_2} \to k_2$.

FIRST TWO DIAGRAMS:

The contribution of the first two diagrams is (after replacing $\varepsilon_{k_2} \to k_2$)

$$i\widetilde{\mathcal{M}}^{ab}_{12\mu\nu}\varepsilon^{\mu}_{k_1}k_2^{\nu} = (-ig)^2 \;\; \bar{v}_{p_1} \;\; \left(\not{\varepsilon}_{k_1}t^a \; \frac{i}{\not{p}_2 - \not{k}_2 - m} \;\; \not{k}_2 t^b \;\; + \;\; \not{k}_2 t^b \; \frac{i}{\not{k}_2 - \not{p}_1 - m} \;\; \not{\varepsilon}_{k_1}t^a\right) \;\; u_{p_2}.$$

Since $(\not{p}_2 - m)u_{p_2} = \bar{v}_{p_1}(\not{p}_1 + m) = 0$, we can replace $\not{k}_2 \to \not{k}_2 - (\not{p}_2 - m)$ and $\not{k}_2 \to \not{k}_2 - (\not{p}_2 + m)$ in the first and second term respectively, to cancel the denominators:

$$\begin{aligned}
i\widetilde{\mathcal{M}}^{ab}_{12\mu\nu}\varepsilon^{\mu}_{k_1}k_2^{\nu} &= (-ig)^2 \;\; \bar{v}_{p_1} \;\; \left(\not{\varepsilon}_{k_1}t^a(-i)t^b \;\; + \;\; t^b i\not{\varepsilon}_{k_1}t^a\right) \;\; u_{p_2} \\
&= -i(-ig)^2 \;\; \bar{v}_{p_1} \;\; \not{\varepsilon}_{k_1}(t^a t^b - t^b t^a) \;\; u_{p_2} = (-ig)^2 \;\; \bar{v}_{p_1} \;\; \left(-i\not{\varepsilon}_{k_1}[t^a, t^b]\right) \;\; u_{p_2} \\
&= (-ig)^2 f^{abc} \;\; \bar{v}_{p_1}\left(\not{\varepsilon}_{k_1}t_c\right)u_{p_2},
\end{aligned}$$

where we used $[t_a, t_b] = if^{abc}t_c$ from section 2.1.

THIRD DIAGRAM:

Replacing $\varepsilon_{k_2} \to k_2$ also in the third diagram, we find

$$\begin{aligned}
i\widetilde{\mathcal{M}}^{ab}_{3\mu\nu} \; \varepsilon^{\mu}_{k_1}k_2^{\nu} &= (ig^2) \;\; \bar{v}_{p_1} \;\; (\gamma_\rho t_c) \;\; \frac{-i}{k^2} \\
&\quad \cdot f^{abc}\left(\left(\varepsilon_{k_1}\cdot k_2\right)(k_2 - k_1)^\rho + \varepsilon^{\rho}_{k_1}\left(k_2\cdot(k_1 - k)\right) + k_2^{\rho}\left(\varepsilon_{k_1}\cdot(k - k_2)\right)\right) \;\; u_{p_2}.
\end{aligned}$$

The bracket contains three terms. The second one yields, plugging in the momentum conservation identiy $k_2 = -k - k_1$,

$$\varepsilon^{\rho}_{k_1}\left(k_2\cdot(k_1 - k)\right) = \varepsilon^{\rho}_{k_1}\left((-k - k_1)\cdot(k_1 - k)\right) = \varepsilon^{\rho}_{k_1}(k^2 - k_1^2).$$

The sum of the first and the third term gives us

$$\begin{aligned}
&\left(\varepsilon_{k_1}\cdot k_2\right)(k_2 - k_1)^\rho + k_2^{\rho}\left(\varepsilon_{k_1}\cdot(k - k_2)\right) \\
&\quad = \left(\varepsilon_{k_1}\cdot(k + k_1)\right)(k + 2k_1)^\rho - (k + k_1)^\rho\left(\varepsilon_{k_1}\cdot(2k + k_1)\right) \\
&\quad = \varepsilon_{k_1}\cdot\left((k + k_1)(k + 2k_1)^\rho - (k + k_1)^\rho(2k + k_1)\right) \\
&\quad = \varepsilon_{k_1}\cdot\left(\left(kk^\rho + 2kk_1^\rho + k_1k^\rho + 2k_1k_1^\rho\right) - \left(2kk^\rho + k_1k^\rho + 2kk_1^\rho + k_1k_1^\rho\right)\right) \\
&\quad = \varepsilon_{k_1}\cdot\left(\left(k_1k_1^\rho\right) - \left(kk^\rho\right)\right) = \left(\varepsilon_{k_1}\cdot k_1\right)k_1^\rho - \left(\varepsilon_{k_1}\cdot k\right)k^\rho.
\end{aligned}$$

Since gauge bosons are massless and since we consider them to be on-shell (which is also demanded by Cutkosky rules), we can use $k_1^2 = 0$ and we find

$$i\widetilde{\mathcal{M}}^{ab}_{3\mu\nu} \; \varepsilon^{\mu}_{k_1}k_2^{\nu} = (ig^2) \;\; \bar{v}_{p_1} \;\; (\gamma_\rho t_c) \;\; \frac{-i}{k^2} \;\; f^{abc} \;\; \left(\varepsilon^{\rho}_{k_1}k^2 - \left(\varepsilon_{k_1}\cdot k\right)k^\rho + \left(\varepsilon_{k_1}\cdot k_1\right)k_1^\rho\right) \;\; u_{p_2}.$$

Using, again, $(\not{p}_2 - m)u_{p_2} = \bar{v}_{p_1}(\not{p}_1 + m) = 0$, the k^ρ of the second term in the bracket yields, together with the spinors and γ_ρ,

$$-\bar{v}_{p_1}\not{k}u_{p_2} = \bar{v}_{p_1}(\not{p}_1 + \not{p}_2)u_{p_2} = \bar{v}_{p_1}(-m + m)u_{p_2} = 0,$$

where we used $-k = p_1 + p_2$ from momentum conservation. Thus, only the following two terms are left:

$$
\begin{aligned}
i\widetilde{\mathcal{M}}^{ab}_{3\mu\nu}\ \varepsilon^{\mu}_{k_1} k_2^{\nu} &= (ig^2)\ \ \bar{v}_{p_1}\ \ (\gamma_\rho t_c)\ \ \frac{-i}{k^2}\ \ f^{abc}\ \ \left(\varepsilon^{\rho}_{k_1} k^2 + (\varepsilon_{k_1}\cdot k_1)k_1^{\rho}\right)\ \ u_{p_2} \\
&= (ig^2)\ \ \bar{v}_{p_1}\ \ (\gamma_\rho t_c)\ \ \frac{-i}{k^2}\ \ f^{abc}\ \ \left(\varepsilon^{\rho}_{k_1} k^2\right)\ \ u_{p_2} \\
&\quad + (ig^2)\ \ \bar{v}_{p_1}\ \ (\gamma_\rho t_c)\ \ \frac{-i}{k^2}\ \ f^{abc}\ \ \left((\varepsilon_{k_1}\cdot k_1)k_1^{\rho}\right)\ \ u_{p_2} \\
&= -\underbrace{i(ig^2)}_{=(-ig)^2} f^{abc}\ \ \bar{v}_{p_1}(\not\varepsilon_{k_1} t_c)u_{p_2}\ \ -\frac{i(ig^2)}{k^2} f^{abc}\ \ \bar{v}_{p_1}(\not k_1 t_c)u_{p_2}\ \ (\varepsilon_{k_1}\cdot k_1).
\end{aligned}
$$

ALL THREE DIAGRAMS COMBINED:

The first term of the third diagram precisely cancels the contributions from the first two diagrams. Only the second term of the third diagrams remains:

$$
i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{k_1} k_2^{\nu} = \ i\widetilde{\mathcal{M}}^{ab}_{12\mu\nu}\varepsilon^{\mu}_{k_1} k_2^{\nu} + \ i\widetilde{\mathcal{M}}^{ab}_{3\mu\nu}\ \varepsilon^{\mu}_{k_1} k_2^{\nu} = -\frac{i(ig^2)}{k^2} f^{abc}\ \ \bar{v}_{p_1}(\not k_1 t_c)u_{p_2}\ \ (\varepsilon_{k_1}\cdot k_1).
$$

So far, this is true for any ε_{k_1}. If $\varepsilon_{k_1}\cdot k_1 = 0$, everything vanishes, which is exactly what we wanted to proof.

18.3.6 Terms that are only Cancelled by Ghosts

Now we consider the terms that do not cancel by themselves. Apart from the terms with physical polarizations only (which we abbreviate by "$+\cdots$"), we found in (>18.3.5) that only the following terms remain:

$$
\begin{aligned}
&\frac{1}{2}\left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\right)\eta^{\mu\rho}\eta^{\nu\sigma}\delta_{ac}\delta_{bd}\left(i\widehat{\mathcal{M}}^{cd}_{\rho\sigma}\right) \\
&\quad= \frac{1}{2}\left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\right)\left(\varepsilon^{\mu}_{-k_1}\varepsilon^{\rho}_{+k_1}\varepsilon^{\nu}_{+k_2}\varepsilon^{\sigma}_{-k_2} + \varepsilon^{\mu}_{+k_1}\varepsilon^{\rho}_{-k_1}\varepsilon^{\nu}_{-k_2}\varepsilon^{\sigma}_{+k_2} + \cdots\right)\delta_{ac}\delta_{bd}\left(i\widehat{\mathcal{M}}^{cd}_{\rho\sigma}\right) \\
&\quad= \frac{1}{2}\left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{-k_1}\varepsilon^{\nu}_{+k_2}\right)\left(i\widehat{\mathcal{M}}^{ab}_{\rho\sigma}\varepsilon^{\rho}_{+k_1}\varepsilon^{\sigma}_{-k_2}\right) + \frac{1}{2}\left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{+k_1}\varepsilon^{\nu}_{-k_2}\right)\left(i\widehat{\mathcal{M}}^{ab}_{\rho\sigma}\varepsilon^{\rho}_{-k_1}\varepsilon^{\sigma}_{+k_2}\right) + \cdots.
\end{aligned}
$$

Consider the first of those brackets and plug in the formula for $\varepsilon^{\nu}_{+k_2}$ from (>18.3.3). Then, we arrive at a structure for which we found an explicit formula at the very end of (>18.3.5):

$$
\begin{aligned}
i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{-k_1}\varepsilon^{\nu}_{+k_2} &= \frac{1}{\sqrt{2}|\vec{k}_2|}\, i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{-k_1} k_2^{\nu} = \frac{1}{\sqrt{2}|\vec{k}_2|}\left(\frac{-i(ig^2)}{k^2} f^{abc}\ \ \bar{v}_{p_1}(\not k_1 t_c)u_{p_2}\ \ (\varepsilon_{-k_1}\cdot k_1)\right) \\
&= \frac{-i(ig^2)}{k^2} f^{abc}\ \ \bar{v}_{p_1}(\not k_1 t_c)u_{p_2}\ \ \frac{|\vec{k}_1|}{|\vec{k}_2|},
\end{aligned}
$$

where we used

$$
\varepsilon_{-k_1}\cdot k_1 = \frac{1}{\sqrt{2}|\vec{k}_1|}\left((k_1^0)^2 + \vec{k}_1^2\right) = \frac{2|\vec{k}_1|^2}{\sqrt{2}|\vec{k}_1|}.
$$

As already stated in (>18.3.2), we can get $\widehat{\mathcal{M}}$ from $\widetilde{\mathcal{M}}$ by the replacements $k_{1,2} \to -k_{1,2}$ and $\bar{v}_{p_1} \to \bar{u}_{p_2'}$ and $u_{p_2} \to v_{p_1'}$. Thereby,

$$
i\widehat{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{-k_1}\varepsilon^{\nu}_{+k_2} = \frac{-i(ig^2)}{k^2} f^{abc}\ \ \bar{u}_{p_2'}(-\not k_1 t_c)v_{p_1'}\ \ \frac{|\vec{k}_1|}{|\vec{k}_2|}.
$$

Now, only two contractions are left:

$$
i\widetilde{\mathcal{M}}_{\mu\nu}\varepsilon^{\mu}_{+k_1}\varepsilon^{\nu}_{-k_2} = i\widetilde{\mathcal{M}}_{\mu\nu}\varepsilon^{\mu}_{-k_1}\varepsilon^{\nu}_{+k_2}\Big|_{k_1\leftrightarrow k_2} = \frac{-i(ig^2)}{k^2} f^{abc}\ \ \bar{v}_{p_1}(\not k_2 t_c)u_{p_2}\ \ \frac{|\vec{k}_2|}{|\vec{k}_1|},
$$

$$i\widehat{\mathcal{M}}_{\rho\sigma}\varepsilon^{\rho}_{+k_1}\varepsilon^{\sigma}_{-k_2} = i\widehat{\mathcal{M}}_{\rho\sigma}\varepsilon^{\rho}_{-k_1}\varepsilon^{\sigma}_{+k_2}\Big|_{k_1\leftrightarrow k_2} = \frac{-i(ig^2)}{k^2} f^{abc} \quad \bar{u}_{p_2'}(-\not{k}_2 t_c) v_{p_1'} \quad \frac{|\vec{k}_2|}{|\vec{k}_1|}.$$

Picking those four contributions up, we find (we write the contributions of the physical polarizations simply as "$+\cdots$" again)

$$\begin{aligned}
&\frac{1}{2}\left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\right)\eta^{\mu\rho}\eta^{\nu\sigma}\delta_{ac}\delta_{bd}\left(i\widehat{\mathcal{M}}^{cd}_{\rho\sigma}\right)\\
&\quad = \frac{1}{2}\left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{-k_1}\varepsilon^{\nu}_{+k_2}\right)\left(i\widehat{\mathcal{M}}^{ab}_{\rho\sigma}\varepsilon^{\rho}_{+k_1}\varepsilon^{\sigma}_{-k_2}\right) + \frac{1}{2}\left(i\widetilde{\mathcal{M}}^{ab}_{\mu\nu}\varepsilon^{\mu}_{+k_1}\varepsilon^{\nu}_{-k_2}\right)\left(i\widehat{\mathcal{M}}^{ab}_{\rho\sigma}\varepsilon^{\rho}_{-k_1}\varepsilon^{\sigma}_{+k_2}\right) + \cdots\\
&\quad = \frac{1}{2}\left(\frac{-i(ig^2)}{k^2} f^{abc} \quad \bar{v}_{p_1}(\not{k}_1 t_c) u_{p_2}\right)\left(\frac{-i(ig^2)}{k^2} f^{abd} \quad \bar{u}_{p_2'}(-\not{k}_2 t_d) v_{p_1'}\right)\\
&\quad\quad + \frac{1}{2}\left(\frac{-i(ig^2)}{k^2} f^{abc} \quad \bar{v}_{p_1}(-\not{k}_2 t_c) u_{p_2}\right)\left(\frac{-i(ig^2)}{k^2} f^{abd} \quad \bar{u}_{p_2'}(\not{k}_1 t_d) v_{p_1'}\right) + \cdots\\
&\quad = \left(\frac{-i(ig^2)}{k^2} f^{abc} \quad \bar{v}_{p_1}(\not{k}_1 t_c) u_{p_2}\right)\left(\frac{-i(ig^2)}{k^2} f^{abd} \quad \bar{u}_{p_2'}(-\not{k}_2 t_d) v_{p_1'}\right) + \cdots.
\end{aligned}$$

In the last step, we set the two term that were added in this expression (apart from "$+\cdots$") equal. This is valid, since

$$\bar{v}_{p_1}(\not{k}_1 + \not{k}_2) u_{p_2} = \bar{v}_{p_1}(\not{p}_1 + \not{p}_2) u_{p_2} = 0 \qquad \Leftrightarrow \qquad \bar{v}_{p_1}(\not{k}_1) u_{p_2} = -\bar{v}_{p_1}(\not{k}_2) u_{p_2},$$

$$\bar{u}_{p_2'}(\not{k}_1 + \not{k}_2) v_{p_1'} = \bar{u}_{p_2'}(\not{p}_1' + \not{p}_2') v_{p_1'} = 0 \qquad \Leftrightarrow \qquad \bar{u}_{p_2'}(\not{k}_1) v_{p_1'} = -\bar{u}_{p_2'}(\not{k}_2) v_{p_1'},$$

where we used $(\not{p} - m)u_p = \bar{u}_p(\not{p} - m) = (\not{p} + m)v_p = \bar{v}_p(\not{p} + m) = 0$.

18.3.7 Using the Ghost Diagram to Cancel the Surplus Terms

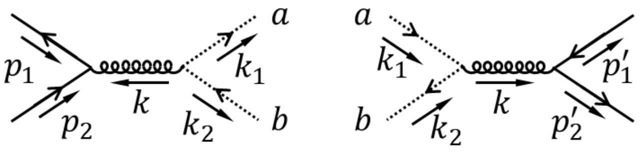
The same we did with the amplitude $\mathcal{M}$, we can also do with the amplitude of the ghost diagram $\mathcal{M}_{\text{gh}}$. By exactly the same steps as in (>18.3.1), we get from

$$i\mathcal{M}_{\text{gh}} = -\int d^4\bar{q}\,\left(i\widetilde{\mathcal{M}}^{ab}_{\text{gh}}\right)\frac{i\delta_{ac}}{k_1^2}\frac{i\delta_{bd}}{k_2^2}\left(i\widehat{\mathcal{M}}^{cd}_{\text{gh}}\right)$$

to

$$2\,\mathrm{Im}\,\mathcal{M}_{\text{gh}} = -\int d\phi\,\left(i\widetilde{\mathcal{M}}^{ab}_{\text{gh}}\right)\delta_{ac}\delta_{bd}\left(i\widehat{\mathcal{M}}^{cd}_{\text{gh}}\right) = -\int d\phi\,\left(i\widetilde{\mathcal{M}}^{ab}_{\text{gh}}\right)\left(i\widehat{\mathcal{M}}^{ab}_{\text{gh}}\right)$$

where $d\phi$ is defined in (>18.3.1). Note the minus sign: It comes from the loop of the ghosts; since ghosts are anticommuting particles, their loops require this minus sign according to Feynman rules (just as for fermions). Obviously, this loop only exists when the two parts $\widetilde{\mathcal{M}}^{ab}_{\text{gh}}$ and $\widehat{\mathcal{M}}^{ab}_{\text{gh}}$ are attached to each other – therefore, is cannot be absorbed in neither $\widetilde{\mathcal{M}}^{ab}_{\text{gh}}$ nor $\widehat{\mathcal{M}}^{ab}_{\text{gh}}$ alone.



$\widetilde{\mathcal{M}}^{ab}_{\text{gh}}$ $\qquad\qquad$ $\widehat{\mathcal{M}}^{ab}_{\text{gh}}$

Using Feynman rules, we find

$$i\widetilde{\mathcal{M}}_{\text{ghost}} = \bar{v}_{p_1} \quad (-ig\gamma^{\mu}t^d) \quad u_{p_2} \quad \frac{-i\eta_{\mu\nu}\delta_{dc}}{k^2} \quad (-gf^{abc}k_1^{\nu}) = \frac{-i(ig^2)}{k^2} f^{abc} \quad \bar{v}_{p_1}(\not{k}_1 t^c) u_{p_2},$$

$$i\widehat{\mathcal{M}}_{\text{ghost}} = \bar{u}_{p_2'} \ \ (-ig\gamma^\mu t^d) \ \ v_{p_1'} \ \ \frac{-i\eta_{\mu\nu}\delta_{dc}}{k^2} \ \ \left(-gf^{abc}(-k_2^\nu)\right) = \frac{-i(ig^2)}{k^2} f^{abc} \ \ \bar{u}_{p_2'}(-\not{k}_2 t^c) v_{p_1'}.$$

Obviously, those terms are exactly the same as the surplus terms from (>18.3.6). Due to the minus sign because of the ghost loop, they cancel. Let's pick up all our results:

$$\begin{aligned}
2\,\mathrm{Im}\,\mathcal{M} &= \frac{1}{2}\int d\phi\ \left(i\widetilde{\mathcal{M}}_{\mu\nu}^{ab}\right) \eta^{\mu\rho}\eta^{\nu\sigma}\delta_{ac}\delta_{bd}\ \left(i\widehat{\mathcal{M}}_{\rho\sigma}^{cd}\right) \\
&= \int d\phi\ \left(\left(\frac{-i(ig^2)}{k^2} f^{abc} \ \ \bar{v}_{p_1}(\not{k}_1 t_c) u_{p_2}\right)\left(\frac{-i(ig^2)}{k^2} f^{abd} \ \ \bar{u}_{p_2'}(-\not{k}_2 t_d) v_{p_1'}\right) + \cdots\right),
\end{aligned}$$

$$\begin{aligned}
2\,\mathrm{Im}\,\mathcal{M}_{\text{gh}} &= -\int d\phi\ \left(i\widetilde{\mathcal{M}}_{\text{gh}}^{ab}\right) \delta_{ac}\delta_{bd}\ \left(i\widehat{\mathcal{M}}_{\text{gh}}^{cd}\right) \\
&= -\int d\phi\ \left(\frac{-i(ig^2)}{k^2} f^{abc} \ \ \bar{v}_{p_1}(\not{k}_1 t^c) u_{p_2}\right)\left(\frac{-i(ig^2)}{k^2} f^{abc} \ \ \bar{u}_{p_2'}(-\not{k}_2 t^c) v_{p_1'}\right),
\end{aligned}$$

where the "$+\cdots$" in $2\,\mathrm{Im}\,\mathcal{M}$ stand for terms with physical polarizations only. Therefore,

$$2\,\mathrm{Im}\,\mathcal{M}_{\text{tot}} = 2\,\mathrm{Im}\big(\mathcal{M} + \mathcal{M}_{\text{gh}}\big) = \int d\phi\ (\text{only physical polarization terms}).$$

18.3.8 Cancellation of Unphysical Polarizations in QED

Any amplitude $\mathcal{M}(p) = \mathcal{M}_\mu(p)\,\varepsilon_{\lambda p}^\mu$ with an external photon with polarization $\varepsilon_{\lambda p}^\mu$ yields a cross section proportional to

$$\sum_{\lambda=1,2} |\mathcal{M}|^2 = \sum_{\lambda=1,2} \mathcal{M}_\mu(p)\,\mathcal{M}_\nu^*(p)\,\varepsilon_{\lambda p}^\mu \varepsilon_{\lambda p}^\nu,$$

where the sum covers only physical polarizations $\lambda = 1, 2$. We saw in (>6.3.1) that we can replace $\sum_\lambda \varepsilon_{\lambda p}^\mu \varepsilon_{\lambda p}^\nu = -\eta^{\mu\nu} + \cdots$, where the further terms "$+\cdots$" vanish by the Ward identity (we have proofed that the Ward identity holds in QED).

To see that this replacement is equivalent to the fact that the unphysical polarizations cancel each other, consider for simplicity the special case, where the momentum of the external photon of interest reads $p^\mu = (p, 0, 0, p)$. Then the two transverse polarization vectors can be chosen to be

$$\varepsilon_{1p}^\mu = (0, 1, 0, 0), \qquad \varepsilon_{2p}^\mu = (0, 0, 1, 0).$$

With this choice, we can explicitly write the sum as

$$\sum_{\lambda=1,2} \mathcal{M}_\mu(p)\,\mathcal{M}_\nu^*(p)\,\varepsilon_{\lambda p}^\mu \varepsilon_{\lambda p}^\nu = |\mathcal{M}_1(p)|^2 + |\mathcal{M}_2(p)|^2.$$

Also, for this choice of the momentum p^μ, the Ward identity reads

$$0 = p^\mu \mathcal{M}_\mu(p) = p\mathcal{M}_0(p) - p\mathcal{M}_3(p) \qquad \Leftrightarrow \qquad \mathcal{M}_0 = \mathcal{M}_3.$$

Thus, we can add a zero to the sum above as follows:

$$\begin{aligned}
\sum_{\lambda=1,2} \mathcal{M}_\mu(p)\,\mathcal{M}_\nu^*(p)\,\varepsilon_{\lambda p}^\mu \varepsilon_{\lambda p}^\nu &= |\mathcal{M}_1(p)|^2 + |\mathcal{M}_2(p)|^2 + |\mathcal{M}_3(p)|^2 - |\mathcal{M}_0(p)|^2 \\
&= -\eta^{\mu\nu}\mathcal{M}_\mu(p)\,\mathcal{M}_\nu^*(p).
\end{aligned}$$

This is just another way to derive the replacement rule $\sum_\lambda \varepsilon_{\lambda p}^\mu \varepsilon_{\lambda p}^\nu \to -\eta^{\mu\nu}$ (to be honest, we only performed this second derivation for a special case). But in this way, we used the cancellation of the unphysical polarizations in a more explicit way. Or, in other words, only because the unphysical polarizations cancel, the replacement rule $\sum_\lambda \varepsilon_{\lambda p}^\mu \varepsilon_{\lambda p}^\nu \to -\eta^{\mu\nu}$ is valid.

18.4 The Gauge Boson Self-Energy

18.4.1 The Fermion Loop Diagram

We evaluated the fermion loop diagram $i\Pi^{\mu\nu}(q)$ as a contribution to the photon self-energy already in section 13.1 and 13.3. The only difference in the non-Abelian case is that the vertices receive the additional factor t^a. Since the Feynman rule, that the trace needs to be taken over loops is still valid in the non-Abelian case (and also applies to the symmetry group space), those factors will appear inside a trace.

Therefore, we can simply copy the leading order result from section 13.1:

$$i\Pi'^{\mu\nu}_{ab} \overset{\text{LO}}{=} i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi_2(q^2)\ \operatorname{Tr} t^a t^b,$$

where we can copy the formula for $\Pi(q^2) = \Pi_2(q^2) + \mathcal{O}(g^3)$ from section 13.4,

$$i\Pi_2(q^2) = -\frac{2i\alpha_0}{\pi}\int_0^1 dx\, x(1-x)\left(\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{\Delta} + \mathcal{O}(\epsilon)\right),$$

where $\Delta = -x(1-x)q^2 + m^2$ and $\epsilon = 4 - d$. Let's use that, to leading order, $\alpha_0 = \alpha = g^2/4\pi$ and write it as

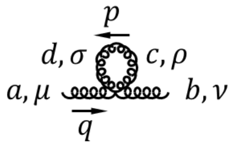
$$i\Pi_2(q^2) = -\frac{8ig^2}{(4\pi)^2}\underbrace{\int_0^1 dx\, x(1-x)}_{=1/6}\cdot\frac{2}{\epsilon} + \text{finite} = -\frac{ig^2}{(4\pi)^2}\frac{8}{3\epsilon} + \text{finite}.$$

Recall from section 2.2 that $\operatorname{Tr} t^a t^b = T(R)\,\delta^{ab}$. Also, if we consider n_f species of fermions, there will be n_f of such loop diagrams (there are $n_f = 6$ quarks that can couple to gluons). Their explicit value depends on the fermion mass m, which appears inside the Δ, but obviously, the divergent part is independent of Δ and m and thus independent of the fermion species. Therefore, to include an arbitrary number of fermion species, we can simply include a factor n_f. Thus, we arrive at

$$i\Pi'^{\mu\nu}_{ab} \overset{\text{LO}}{=} i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,T(R)\,\delta_{ab}\left(\frac{-g^2}{(4\pi)^2}\cdot\frac{8n_f}{3\epsilon} + \text{finite}\right).$$

18.4.2 Single-Vertex Gauge Boson Loop Diagram

Using the notations



and the Feynman rules from section 18.1 for the four gauge boson vertex as well as the gauge boson propagator from section 18.2 in the Feynman-'t-Hooft gauge $\xi = 1$, we find that the amputated amplitude of this vertex reads

$$\frac{1}{2}\int d^d\bar{p}\left(\frac{-i\delta^{cd}\eta_{\sigma\rho}}{p^2}\right)\Big(-ig^2\Big(f^{abe}f^{ecd}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ace}f^{ebd}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})$$
$$+ f^{ade}f^{ebc}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\Big)\Big).$$

Using the Kronecker Delta δ^{cd}, we can turn all d's into c's. Then, the first term of the vertex factor vanishes, since f^{abc} is totally antisymmetric and thus $f^{ecc} = 0$. For the other two terms we can use the normalization $f^{acd}f^{bcd} = N\delta^{ab}$ from section 2.2, where $N = 3$, since we have a SU(3) theory here. Also, we contract the Lorentz indices using $\eta_{\sigma\rho}$. Then we find, using $\eta^\mu_\mu = d$,

$$-g^2N(d-1)\delta^{ab}\eta^{\mu\nu}\int d^d\bar{p}\;\frac{1}{p^2}.$$

Performing a Wick rotation $p^0 \to ip^{0E}$ and using the formulas for dimensional regularized integrals in section 12.6, we find for the integral

$$\int d^d\bar{p}\frac{1}{p^2} \sim \int d^d\bar{p}_E\frac{1}{p_E^2} = \int d^d\bar{p}_E\frac{1}{p_E^2+\Delta}\bigg|_{\Delta=0} = \frac{1}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(1-2+\epsilon/2)}{\Gamma(1)}\frac{1}{\Delta^{1-2+\epsilon/2}}\bigg|_{\Delta=0} = 0.$$

Thus, in dimensional regularization, this diagram does not contribute.

18.4.3 Sum of the last three Diagrams

Note that $\tilde{\pi}^{\mu\nu}, \hat{\pi}^{\mu\nu}$ and $\breve{\pi}^{\mu\nu}$ are all linear combinations of $\Gamma(1-d/2)\eta^{\mu\nu}q^2$ and $\Gamma(2-d/2)\eta^{\mu\nu}q^2$ and $\Gamma(2-d/2)q^{\mu\nu}$. Thus, obviously also the sum $\tilde{\pi}^{\mu\nu}+\hat{\pi}^{\mu\nu}+\breve{\pi}^{\mu\nu}$ will be a linear combination of those expressions:

$$\tilde{\pi}^{\mu\nu}+\hat{\pi}^{\mu\nu}+\breve{\pi}^{\mu\nu} = \Gamma(1-d/2)\eta^{\mu\nu}q^2\,\mathcal{A} + \Gamma(2-d/2)\eta^{\mu\nu}q^2\,\mathcal{B} + \Gamma(2-d/2)q^\mu q^\nu\,\mathcal{C},$$

where

$$\begin{aligned}\mathcal{A} &= \tilde{\mathcal{A}}+\hat{\mathcal{A}}+\breve{\mathcal{A}} = x(1-x)\left(\frac{3}{2}(d-1)-\frac{1}{2}d(d-1)-\frac{1}{2}\right) = x(1-x)\cdot\frac{1}{2}(4d-4-d^2)\\ &= x(1-x)\cdot(1-d/2)(d-2)\end{aligned}$$

and thus

$$\begin{aligned}\Gamma(1-d/2)\eta^{\mu\nu}q^2\,\mathcal{A} &= (1-d/2)\Gamma(1-d/2)\eta^{\mu\nu}q^2\,x(1-x)(d-2)\\ &= \Gamma(2-d/2)\eta^{\mu\nu}q^2\,x(1-x)(d-2).\end{aligned}$$

Thus, this term can be combined with the second one, the $\mathcal{B}$-term. Plugging also in $\mathcal{B} = \widetilde{\mathcal{B}}+\widehat{\mathcal{B}}+0$, we find

$$\begin{aligned}\tilde{\pi}^{\mu\nu}+\hat{\pi}^{\mu\nu}+\breve{\pi}^{\mu\nu} &= \Gamma(2-d/2)\eta^{\mu\nu}q^2\;\left(x(1-x)(d-2)+\mathcal{B}\right)+\mathcal{C}\text{-term}\\ &= \Gamma(2-d/2)\eta^{\mu\nu}q^2\underbrace{\left(x(1-x)(d-2)+\frac{1}{2}(2-x)^2+\frac{1}{2}(1+x)^2-(d-1)(1-x)^2\right)}_{=:\mathcal{B}'}\\ &\quad+\mathcal{C}\text{-term}.\end{aligned}$$

Underneath the integral, the only x dependencies come from Δ and $\mathcal{B}'$ (and the $\mathcal{C}$-term, we will consider later). Note, that $\Delta \sim x(1-x)$. When we substitute $x \to 1-x$, Δ will remain the same and also the integral itself will: $\int_0^1 dx \to \int_1^0(-dx) = \int_0^1 dx$. Thus, we can simply plug in $1-x$ for any x in $\mathcal{B}'$. Of course, within each single term of $\mathcal{B}'$, we must replace *all* x by $1-x$, not just some of them. However, terms linear in x have only one x and then we can replace this x as follows:

$$x = \frac{1}{2}x+\frac{1}{2}x \to \frac{1}{2}x+\frac{1}{2}(1-x) = \frac{1}{2}.$$

To get an elegant result, we only replace the factors x which are marked by red color by $1/2$:

$$\begin{aligned}\mathcal{B}' &= x(1-x)(d-2)+\frac{1}{2}(2-x)^2+\frac{1}{2}(1+x)^2-(d-1)(1-x)^2\\ &= (x-x^2)(d-2)+\frac{1}{2}(4-4x+x^2)+\frac{1}{2}(1+2x+x^2)-(d-1)(1-2x+x^2)\\ &= \frac{d}{2}(2x-2x^2-2+4x-2x^2)-2(x-x^2)+\frac{1}{2}(4-4x+x^2)+\frac{1}{2}(1+2x+x^2)\\ &\quad+(1-2x+x^2)\end{aligned}$$

$$= -\frac{d}{2}(2-2x-4x+4x^2)+(1-4x+4x^2)-x+2+\frac{1}{2}$$
$$= -\frac{d}{2}(1-4x+4x^2)+(1-4x+4x^2)+2$$
$$= \left(1-\frac{d}{2}\right)(1-2x)^2+2.$$

Finally,

$$\mathcal{C} = \tilde{\mathcal{C}} + \breve{\mathcal{C}} = -\left(1-\frac{d}{2}\right)(1-2x)^2 - \underbrace{(1+x)(2-x)}_{=2+x(1-x)} + x(1-x) = -\left(1-\frac{d}{2}\right)(1-2x)^2 - 2.$$

Obviously, $\mathcal{C} = -\mathcal{B}'$. Therefore,

$$\begin{aligned}\tilde{\pi}^{\mu\nu} + \hat{\pi}^{\mu\nu} + \breve{\pi}^{\mu\nu} &= \Gamma(2-d/2)\eta^{\mu\nu}q^2\,\mathcal{B}' + \Gamma(2-d/2)q^\mu q^\nu\,\mathcal{C} \\ &= \Gamma(2-d/2)\,(\eta^{\mu\nu}q^2\,\mathcal{B}' + q^\mu q^\nu\,\mathcal{C}) = \Gamma(2-d/2)\,\mathcal{B}'\,(\eta^{\mu\nu}q^2 - q^\mu q^\nu).\end{aligned}$$

18.4.4 Divergent Part of the last three Diagrams

To find the divergent part of the last three diagrams, write them down explicitly:

$$\begin{aligned}\tilde{\Pi}^{\mu\nu}_{ab} + \hat{\Pi}^{\mu\nu}_{ab} + \breve{\Pi}^{\mu\nu}_{ab} &= \frac{i\mu^{4-d}g^2}{(4\pi)^{d/2}}C_A\delta_{ab}\int_0^1 dx\,\frac{1}{\Delta^{2-d/2}}(\tilde{\pi}^{\mu\nu} + \hat{\pi}^{\mu\nu} + \breve{\pi}^{\mu\nu}) \\ &= \frac{i\mu^{4-d}g^2}{(4\pi)^{d/2}}C_A\delta_{ab}(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\int_0^1 dx\,\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\left(\left(1-\frac{d}{2}\right)(1-2x)^2+2\right) \\ &= \frac{ig^2}{(4\pi)^2}\,C_A\delta_{ab}(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\int_0^1 dx\,\frac{\mu^\epsilon\Gamma(\epsilon/2)}{(4\pi)^{-\epsilon/2}\Delta^{\epsilon/2}}\left(\left(1-\frac{d}{2}\right)(1-2x)^2+2\right).\end{aligned}$$

As in (>18.4.1), we can keep only $\Gamma(\epsilon/2) = 2/\epsilon + \text{finite}$ of the fraction in the integral, since $(4\pi)^{-\epsilon/2}\Delta^{\epsilon/2}\mu^\epsilon$ only contributes a factor of 1 to the divergent part:

$$\begin{aligned}&\tilde{\Pi}^{\mu\nu}_{ab} + \hat{\Pi}^{\mu\nu}_{ab} + \breve{\Pi}^{\mu\nu}_{ab} \\ &= \frac{ig^2}{(4\pi)^2}\,C_A\delta_{ab}(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\left(\Gamma(\epsilon/2)\underbrace{\int_0^1 dx\left(\left(1-\frac{d}{2}\right)(1-2x)^2+2\right)}_{=5/3\text{ for }d=4.} + \text{finite}\right) \\ &= i(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\,C_A\delta_{ab}\left(\frac{-g^2}{(4\pi)^2}\left(-\frac{5}{3}\right)\frac{2}{\epsilon} + \text{finite}\right).\end{aligned}$$

18.7 Counter Terms

18.7.1 Counter Lagrangian

We start out with the Lagrangian from section 17.2:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 + \bar{\Psi}(i\not{D} - m)\Psi + \bar{\vartheta}\left(-\partial^\mu D_\mu\right)\vartheta.$$

This Lagrangian is the bare Lagrangian with the bare fields and the bare coupling constant g_0.

FIRST TERM:

Let's start with the first term, using our expansion from (>18.1.3):

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a$$
$$= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) + g_0 f^{abc}\left(\partial_\mu A^a_\nu\right)A^\mu_b A^\nu_c - \frac{1}{4}g_0^2 f^{abc}f^{ade}\,A^b_\mu A^c_\nu A^\mu_d A^\nu_e.$$

The first term of $\mathcal{L}$ thereby can be written as three terms. Let's consider them one by one:

$$-\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) = -\frac{1}{4}Z_3\left(\partial_\mu A^a_{r\nu} - \partial_\nu A^a_{r\mu}\right)\left(\partial^\mu A^\nu_{ra} - \partial^\nu A^\mu_{ra}\right)$$
$$= (1+\delta_3)\cdot\left(-\frac{1}{4}\left(\partial_\mu A^a_{r\nu} - \partial_\nu A^a_{r\mu}\right)^2\right),$$

$$g_0 f^{abc}\left(\partial_\mu A^a_\nu\right)A^\mu_b A^\nu_c = g_0 Z_3^{3/2} f^{abc}\left(\partial_\mu A^a_{r\nu}\right)A^\mu_{rb}A^\nu_{rc} = \left(1+\delta_1^{3g}\right)\cdot g f^{abc}\left(\partial_\mu A^a_{r\nu}\right)A^\mu_{rb}A^\nu_{rc},$$

$$-\frac{1}{4}g_0^2 f^{abc}f^{ade}\,A^b_\mu A^c_\nu A^\mu_d A^\nu_e = -\frac{1}{4}g_0^2 Z_3^2 f^{abc}f^{ade}\,A^b_{r\mu}A^c_{r\nu}A^\mu_{rd}A^\nu_{re}$$
$$= \left(1+\delta_1^{4g}\right)\left(-\frac{1}{4}g^2 f^{abc}f^{ade}\,A^b_{r\mu}A^c_{r\nu}A^\mu_{rd}A^\nu_{re}\right).$$

Thereby, the first term of $\mathcal{L}$ reads

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a$$
$$= -\frac{1}{4}F^a_{r\mu\nu}F^{\mu\nu}_{ra} - \frac{1}{4}\delta_3\left(\partial_\mu A^a_{r\nu} - \partial_\nu A^a_{r\mu}\right)\left(\partial^\mu A^\nu_{ra} - \partial^\nu A^\mu_{ra}\right) + \delta_1^{3g} g f^{abc}\left(\partial_\mu A^a_{r\nu}\right)A^\mu_{rb}A^\nu_{rc}$$
$$-\frac{1}{4}\delta_1^{4g} g^2 f^{abc}f^{ade}\,A^b_{r\mu}A^c_{r\nu}A^\mu_{rd}A^\nu_{re},$$

where $F^a_{r\mu\nu}$ is just equal $F^a_{\mu\nu}$, but with the renormalized fields and g instead of g_0.

SECOND TERM:

Luckily, the second term is a bit simpler:

$$-\frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 = -\frac{1}{2\xi}Z_3\left(\partial^\mu A^a_{r\mu}\right)^2 = (1+\delta_3)\cdot\left(-\frac{1}{2\xi}\left(\partial^\mu A^a_{r\mu}\right)^2\right)$$
$$= -\frac{1}{2\xi}\left(\partial^\mu A^a_{r\mu}\right)^2 - \frac{1}{2\xi}\delta_3\left(\partial^\mu A^a_{r\mu}\right)^2.$$

THIRD TERM:

The third term is complicated enough, again, to be worth consider its subparts,

$$\bar\Psi(i\not{D} - m)\Psi = i\bar\Psi\not\partial\Psi - g_0\bar\Psi A^a_\mu t^a\Psi - m_0\bar\Psi\Psi,$$

and consider them one by one:

$$i\bar\Psi\not\partial\Psi = iZ_2\bar\Psi_r\not\partial\Psi_r = (1+\delta_2)\cdot i\bar\Psi_r\not\partial\Psi_r,$$

$$-g_0\bar\Psi A^a_\mu t^a\Psi = -g_0 Z_2 Z_3^{1/2}\bar\Psi_r A^a_{r\mu}t^a\Psi_r = (1+\delta_1)\cdot\left(-g\bar\Psi_r A^a_{r\mu}t^a\Psi_r\right),$$

$$-m_0\bar\Psi\Psi = -m_0 Z_2\bar\Psi_r\Psi_r = -(m+\delta_m)\bar\Psi_r\Psi_r.$$

Thus,

$$\bar\Psi(i\not{D} - m)\Psi = \bar\Psi_r(i\not{D}_r - m)\Psi_r + i\delta_2\bar\Psi_r\not\partial\Psi_r - g\delta_1\bar\Psi_r A^a_{r\mu}t^a\Psi_r - \delta_m\bar\Psi_r\Psi_r$$
$$= \bar\Psi_r(i\not{D}_r - m)\Psi_r + \bar\Psi_r(i\delta_2\not\partial - \delta_m)\Psi_r - g\delta_1\bar\Psi_r A^a_{r\mu}t^a\Psi_r,$$

where $D_{r\mu}$ is just equal D_μ, but with the renormalized fields and g instead of g_0.

FOURTH TERM:

Recall from section 17.2, that the t^a inside D_μ needs to be chosen in the adjoint representation: $(t^a)_{bc} = -if^{abc}$. Then,

$$\begin{aligned}\bar{\vartheta}(-\partial^\mu D_\mu)\vartheta &= \bar{\vartheta}^a(-\partial^\mu D_\mu^{ab})\vartheta^b = \bar{\vartheta}^a\left(-\partial^\mu\left(\delta^{ab}\partial_\mu + ig_0A_\mu^c(t^c)_{ab}\right)\right)\vartheta^b \\ &= \bar{\vartheta}^a\left(-\partial^\mu\left(\delta^{ab}\partial_\mu + g_0A_\mu^c f^{cab}\right)\right)\vartheta^b = \bar{\vartheta}^a\left(-\partial^\mu\left(\delta^{ab}\partial_\mu - g_0A_\mu^c f^{acb}\right)\right)\vartheta^b \\ &= \bar{\vartheta}^a\left(-\delta^{ab}\Box + g_0\partial_\mu A_\mu^c f^{acb}\right)\vartheta^b = -\delta^{ab}\bar{\vartheta}^a\Box\vartheta^b + g_0\bar{\vartheta}^a\partial_\mu A_\mu^c f^{acb}\vartheta^b.\end{aligned}$$

Having rewritten this term into a more practical form, we can now renormalize:

$$\begin{aligned}\bar{\vartheta}(-\partial^\mu D_\mu)\vartheta &= -Z_4\delta^{ab}\bar{\vartheta}_r^a\Box\vartheta_r^b + g_0Z_4Z_3^{1/2}\bar{\vartheta}_r^a\partial_\mu A_{r\mu}^c f^{acb}\vartheta_r^b \\ &= (1+\delta_4)\cdot(-\delta^{ab}\bar{\vartheta}_r^a\Box\vartheta_r^b) + \left(1+\delta_1^\vartheta\right)\cdot g\bar{\vartheta}_r^a\partial_\mu A_{r\mu}^c f^{acb}\vartheta_r^b \\ &= \bar{\vartheta}_r\left(-\partial^\mu D_{r\mu}\right)\vartheta_r - \delta_4\bar{\vartheta}_r^a\Box\vartheta_r^a + g\delta_1^\vartheta\bar{\vartheta}_r^a\partial_\mu A_{r\mu}^c f^{acb}\vartheta_r^b,\end{aligned}$$

where $D_{r\mu}$ is just equal D_μ, but with the renormalized fields and g instead of g_0.

FINAL RESULT:

Picking up all results, we can write the Lagrangian as a renormalized Lagrangian $\mathcal{L}_r$ of the same form as the old one, but with the renormalized fields and g instead of g_0, and a bunch of counter terms $\mathcal{L}_{\text{ct}}$:

$$\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{\text{ct}},$$

where

$$\begin{aligned}\mathcal{L}_{\text{ct}} = &-\frac{1}{4}\delta_3\left(\partial_\mu A_{r\nu}^a - \partial_\nu A_{r\mu}^a\right)^2 + \bar{\Psi}_r(i\delta_2\partial\!\!\!/ - \delta_m)\Psi_r - \delta_4\bar{\vartheta}_r^a\Box\vartheta_r^a - \frac{1}{2\xi}\delta_3\left(\partial^\mu A_{r\mu}^a\right)^2 \\ &- g\delta_1\bar{\Psi}_r A_{r\mu}^a t^a\Psi_r + \delta_1^{3g} gf^{abc}\left(\partial_\mu A_{r\nu}^a\right)A_{rb}^\mu A_{rc}^\nu - \frac{1}{4}\delta_1^{4g}g^2 f^{abc}f^{ade}\,A_{r\mu}^b A_{r\nu}^c A_{rd}^\mu A_{re}^\nu \\ &+ g\delta_1^\vartheta\bar{\vartheta}_r^a\partial_\mu A_{r\mu}^c f^{acb}\vartheta_r^b.\end{aligned}$$

18.7.2 Value of the Counter Term Parameters

VERTEX COUNTER TERM:

From section 17.6 we know that the infinite parts of the vertex correction read

$$\frac{ig^3}{(4\pi)^2}t^a\gamma^\mu\frac{2}{\epsilon}\left(C_2 - \frac{1}{2}C_A + \frac{3}{2}C_A\right) = igt^a\gamma^\mu\,\frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}(2C_2 + 2C_A).$$

We want to cancel this infinity with the counter vertex $igt^a\gamma^\mu\delta_1$, thus δ_1 must be

$$\delta_1 = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}(-2C_2 - 2C_A).$$

For a general gauge ξ, one can show that

$$\delta_1 = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}\left(-2C_2 - 2C_A + 2(1-\xi)C_2 + \frac{1}{2}(1-\xi)C_A\right).$$

ELECTRON SELF-ENERGY:

From section 17.5 we know that the infinite parts of the electron self-energy read

$$\frac{ig^2}{(4\pi)^2}\frac{1}{\epsilon}C_2(2p\!\!\!/ - 8m).$$

We want to cancel this infinity with the counter fermion propagator $i(\delta_2 p\!\!\!/ - \delta_m)$, thus δ_2 and δ_m must be (here now for a general gauge ξ)

$$\delta_2 = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}(-2C_2 + 2(1-\xi)C_2), \qquad \delta_m = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}(-8C_2 m).$$

GAUGE BOSON SELF-ENERGY:

From section 17.4, we know that the infinite parts of the gauge boson self-energy read

$$i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\delta_{ab}\frac{-g^2}{(4\pi)^2}\frac{1}{\epsilon}\left(\frac{8n_f}{3}T - 2\cdot\frac{5}{3}C_A\right).$$

We want to cancel this infinity with the counter gauge boson propagator $-i\delta_3(q^2\eta^{\mu\nu} - q^\mu q^\nu)$, thus δ_3 must be (here now for a general gauge ξ)

$$\delta_3 = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}\left(\frac{10}{3}C_A - \frac{8n_f}{3}T + (1-\xi)C_A\right).$$

18.8 Asymptotic Freedom

18.8.1 The β Function

In (>17.6.3) we used the regularization rule $g = Z^{-1}\mu^{(d-4)/2}g_0$ from (>17.6.1) to show that

$$\mu\frac{dg}{d\mu} = -\frac{\epsilon}{2}g + \mathcal{O}(g^3).$$

Thus,

$$\mu\frac{d}{d\mu}f\big(g(\mu)\big) = \mu\frac{dg}{d\mu}\frac{\partial f(g)}{dg} = \left(-\frac{\epsilon}{2}g + \mathcal{O}(g^3)\right)\frac{\partial f(g)}{dg}.$$

Using the following formula for the β function in terms of the δ's from section 16.6 (derived there for QED, but still valid in the non-Abelian theory) together with the explicit formulas for the δ's from the very end of section 17.7, we find

$$\begin{aligned}\beta(g) &= g\frac{\epsilon}{2} + g\mu\frac{d}{d\mu}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right) = g\frac{\epsilon}{2} + g\left(-\frac{\epsilon}{2}g\right)\frac{\partial}{dg}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right)\\ &= g\frac{\epsilon}{2} + g\left(-\frac{\epsilon}{2}g\right)\frac{1}{\epsilon}\frac{2g}{(4\pi)^2}\left((-2C_2 - 2C_A) - (-2C_2) - \frac{1}{2}\left(\frac{10}{3}C_A - \frac{8n_f}{3}T\right)\right)\\ &= g\frac{\epsilon}{2} + \frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_A - \frac{4n_f}{3}T\right) \to \frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_A - \frac{4n_f}{3}T\right).\end{aligned}$$

In the last step, we set $\epsilon \to 0$.

18.8.2 The Running Coupling

We can derive the running coupling for a general β function

$$\beta = \frac{1}{2}Cg^3$$

with constant C by the following equation from section 16.7 ($d\lambda \to -dg$):

$$\underbrace{\int_{\wp'=\mu}^{\wp'=\wp} d\ln(\wp'/\mu)}_{=\ln\wp/\mu} = -\int_g^{\bar g} dg'\frac{1}{\beta(g')} = -\frac{2}{C}\int_g^{\bar g} dg'\, g'^{-3} = -\frac{2}{C}\left[-\frac{1}{2}g'^{-2}\right]_g^{\bar g} = \frac{1}{C}\left(\frac{1}{\bar g^2} - \frac{1}{g^2}\right)$$

$$\Leftrightarrow \qquad \bar g^2(\wp,\mu) = \frac{g^2}{1 + g^2 C\ln\wp/\mu}.$$

Thus, for our present case of

$$C = \frac{2}{(4\pi)^2}\left(\frac{11}{3}C_A - \frac{4n_f}{3}T\right),$$

we find

$$\bar{g}^2(\wp, \mu) = \frac{g^2}{1 + g^2 \frac{2}{(4\pi)^2}\left(\frac{11}{3}C_A - \frac{4n_f}{3}T\right)\ln \wp/\mu} = \frac{g^2}{1 + \frac{g^2}{(4\pi)^2}\left(\frac{11}{3}C_A - \frac{4n_f}{3}T\right)\ln \wp^2/\mu^2}.$$

19 The Higgs Mechanism

19.1 The Linear Sigma Model

19.1.1 Minimum of the Potential

To find the minimum of the potential

$$V(\vec{\phi}) = -\frac{\mu^2}{2}\vec{\phi}^2 + \frac{\lambda}{4}\vec{\phi}^4,$$

we need to solve

$$\nabla V(\vec{\phi}) = -\mu^2\vec{\phi} + \lambda\vec{\phi}^2\,\vec{\phi} \stackrel{!}{=} 0.$$

Obviously, there is one extremum at $\vec{\phi} = 0$ and one at

$$\vec{\phi}^2 = \frac{\mu^2}{\lambda} =: \vec{\phi}_0^2.$$

Obviously, there are many vectors $\vec{\phi}_0$ that obey this definition, since it only fixes the length of the vector to $\mu/\sqrt{\lambda}$. Thus, all vectors $\vec{\phi}_0$ lie on an N-dimensional sphere. Since V tends to $+\infty$ for large $\vec{\phi}$, it has a minimum at $\vec{\phi} = \vec{\phi}_0$ and a maximum at $\vec{\phi} = 0$.

To be specific, we chose $\vec{\phi}_0$ not to be an arbitrary vector pointing to that sphere, but one specific of them. It is conventional to choose

$$\vec{\phi}_0 \coloneqq (0, 0, \dots, 0, v), \qquad v \coloneqq \mu/\sqrt{\lambda}.$$

19.1.2 Lagrangian in Terms of Deviations from the Minimum

We want to plug

$$\vec{\phi} = \left(\vec{\pi}(x), v + \sigma(x)\right)$$

into the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu\vec{\phi}\right)^2 + \frac{\mu^2}{2}\vec{\phi}^2 - \frac{\lambda}{4}\vec{\phi}^4.$$

Term by term, we find

$$\frac{1}{2}\left(\partial^\mu\vec{\phi}\right)^2 = \frac{1}{2}(\partial^\mu\vec{\pi})^2 + \frac{1}{2}(\partial^\mu\sigma)^2,$$

$$\begin{aligned}\frac{\mu^2}{2}\vec{\phi}^2 &= \frac{\mu^2}{2}(\vec{\pi}^2 + (v+\sigma)^2) = \frac{\mu^2}{2}(\vec{\pi}^2 + v^2 + 2v\sigma + \sigma^2) = \frac{\mu^2}{2}\left(\vec{\pi}^2 + \frac{\mu^2}{\lambda} + \frac{2\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right)\\ &= \frac{\mu^2}{2}\vec{\pi}^2 + \frac{\mu^4}{2\lambda} + \frac{\mu^3}{\sqrt{\lambda}}\sigma + \frac{\mu^2}{2}\sigma^2,\end{aligned}$$

$$\begin{aligned}-\frac{\lambda}{4}\vec{\phi}^4 &= -\frac{\lambda}{4}(\vec{\pi}^2 + (v+\sigma)^2)^2 = -\frac{\lambda}{4}(\vec{\pi}^4 + 2\vec{\pi}^2(v+\sigma)^2 + (v+\sigma)^4)\\ &= -\frac{\lambda}{4}(\vec{\pi}^4 + 2\vec{\pi}^2(v^2 + 2v\sigma + \sigma^2) + v^4 + 4v^3\sigma + 6v^2\sigma^2 + 4v\sigma^3 + \sigma^4)\\ &= -\frac{\lambda}{4}\left(\vec{\pi}^4 + 2\vec{\pi}^2\left(\frac{\mu^2}{\lambda} + \frac{2\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right) + \frac{\mu^4}{\lambda^2} + \frac{4\mu^3}{\lambda^{3/2}}\sigma + \frac{6\mu^2}{\lambda}\sigma^2 + \frac{4\mu}{\sqrt{\lambda}}\sigma^3 + \sigma^4\right)\\ &= -\frac{\lambda}{4}\vec{\pi}^4 - \frac{\mu^2}{2}\vec{\pi}^2 - \mu\sqrt{\lambda}\vec{\pi}^2\sigma - \frac{\lambda}{2}\vec{\pi}^2\sigma^2 - \frac{\mu^4}{4\lambda} - \frac{\mu^3}{\sqrt{\lambda}}\sigma - \frac{3\mu^2}{2}\sigma^2 - \mu\sqrt{\lambda}\sigma^3 - \frac{\lambda}{4}\sigma^4.\end{aligned}$$

Thus, the whole Lagrangian reads

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial^\mu \vec{\pi})^2 + \frac{1}{2}(\partial^\mu \sigma)^2 + \left(\frac{\mu^4}{2\lambda} - \frac{\mu^4}{4\lambda}\right) + \left(\frac{\mu^3}{\sqrt{\lambda}} - \frac{\mu^3}{\sqrt{\lambda}}\right)\sigma + \left(\frac{\mu^2}{2} - \frac{\mu^2}{2}\right)\vec{\pi}^2 + \left(\frac{\mu^2}{2} - \frac{3\mu^2}{2}\right)\sigma^2 \\ &\quad - \mu\sqrt{\lambda}\,\sigma^3 - \mu\sqrt{\lambda}\,\vec{\pi}^2\sigma - \frac{\lambda}{4}\vec{\pi}^4 - \frac{\lambda}{2}\vec{\pi}^2\sigma^2 - \frac{\lambda}{4}\sigma^4 \\ &= \frac{1}{2}(\partial^\mu \vec{\pi})^2 + \frac{1}{2}(\partial^\mu \sigma)^2 + \frac{\mu^4}{4\lambda} - \mu^2\,\sigma^2 - \mu\sqrt{\lambda}\,\sigma^3 - \mu\sqrt{\lambda}\,\vec{\pi}^2\sigma - \frac{\lambda}{4}\vec{\pi}^4 - \frac{\lambda}{2}\vec{\pi}^2\sigma^2 - \frac{\lambda}{4}\sigma^4.\end{aligned}$$

We can drop the constant term $\mu^4/4\lambda$, since constant terms are of no interest inside Lagrangians.

19.2 Goldstone's Theorem

19.2.1 Proof of Goldston's Theorem

Consider a theory of fields ϕ_i, which we can write as a vector $\vec{\phi} = (\phi_1, \dots, \phi_i)$, with a Lagrangian of the form $\mathcal{L} = T - V$, where T contains all the terms with derivatives and V all terms without any derivatives. Let $\vec{\phi}_0$ minimize V, that is

$$\nabla V(\vec{\phi})\big|_{\vec{\phi}=\vec{\phi}_0} = 0 \qquad \Leftrightarrow \qquad \frac{\partial}{\partial \phi_i} V(\vec{\phi})\Big|_{\vec{\phi}=\vec{\phi}_0} = 0 \;\forall i.$$

Expanding V about this minimum $\vec{\phi}_0$ yields

$$\begin{aligned} V(\vec{\phi}) &= V(\vec{\phi}_0) + \sum_i \underbrace{\frac{\partial}{\partial \phi_i} V(\vec{\phi})\Big|_{\vec{\phi}=\vec{\phi}_0}}_{=0} (\phi_i - \phi_{0i}) \\ &\qquad + \frac{1}{2}\sum_{ij} \underbrace{\frac{\partial}{\partial \phi_i}\frac{\partial}{\partial \phi_j} V(\vec{\phi})\Big|_{\vec{\phi}=\vec{\phi}_0}}_{=:M_{ij}^2} (\phi_i - \phi_{0i})(\phi_j - \phi_{0j}) \\ &= V(\vec{\phi}_0) + \sum_{ij} \frac{\partial}{\partial \phi_i}\frac{\partial}{\partial \phi_j} V(\vec{\phi})\Big|_{\vec{\phi}=\vec{\phi}_0} (\phi_i - \phi_{0i})(\phi_j - \phi_{0j}). \end{aligned}$$

The coefficient M_{ij}^2 is a symmetric matrix whose eigenvalues give the masses of the fields (since they are the prefactors of the squared fields).

A general continuous symmetry transformation has the form

$$\vec{\phi} \to \vec{\phi} + \epsilon \vec{\Delta}(\vec{\phi}),$$

where ϵ is an infinitesimal parameter and $\vec{\Delta}(\vec{\phi})$ a function of all fields ϕ_i (that is, $\vec{\Delta}$ is chosen such that the Lagrangian is unchanged; the fields on the other hand are arbitrary). If a Lagrangian is symmetric under this transformation, it is symmetric under this transformation regardless of what kind of function $\vec{\phi}(x)$ is. In particular, it must also be symmetric under this transformation, if the fields are constant. Then, the terms with derivatives in the Lagrangian vanish, $T = 0$, and the potential alone must be invariant under this transformation:

$$\begin{aligned} & V(\vec{\phi}) \overset{!}{=} V\left(\vec{\phi} + \epsilon\vec{\Delta}(\vec{\phi})\right) \\ \Leftrightarrow \qquad & \epsilon\Delta_i(\vec{\phi}) \frac{V\left(\vec{\phi} + \epsilon\vec{\Delta}(\vec{\phi})\right) - V(\vec{\phi})}{\epsilon\Delta_i(\vec{\phi})} \overset{!}{=} 0 \\ \Leftrightarrow \qquad & \Delta_i(\vec{\phi}) \frac{\partial V(\vec{\phi})}{\partial \phi_i} \overset{!}{=} 0 \end{aligned}$$

$$\Rightarrow \qquad 0 \stackrel{!}{=} \frac{\partial}{\partial \phi_j}\left(\Delta_i(\vec{\phi})\frac{\partial V(\vec{\phi})}{\partial \phi_i}\right) = \frac{\partial \Delta_i(\vec{\phi})}{\partial \phi_j}\frac{\partial V(\vec{\phi})}{\partial \phi_i} + \Delta_i(\vec{\phi})\frac{\partial}{\partial \phi_j}\frac{\partial}{\partial \phi_i}V(\vec{\phi}).$$

If we now plug in $\vec{\phi} = \vec{\phi}_0$, we find

$$\left.\frac{\partial \Delta_i(\vec{\phi})}{\partial \phi_j}\right|_{\vec{\phi}=\vec{\phi}_0} \cdot \underbrace{\left.\frac{\partial V(\vec{\phi})}{\partial \phi_i}\right|_{\vec{\phi}=\vec{\phi}_0}}_{=0} + \Delta_i(\vec{\phi}_0)\underbrace{\left.\frac{\partial}{\partial \phi_j}\frac{\partial}{\partial \phi_i}V(\vec{\phi})\right|_{\vec{\phi}=\vec{\phi}_0}}_{=:M_{ji}^2} \stackrel{!}{=} 0 \qquad \Leftrightarrow \qquad M_{ji}^2\,\Delta_i(\vec{\phi}_0) \stackrel{!}{=} 0.$$

If the transformation $\vec{\phi} \to \vec{\phi} + \epsilon\vec{\Delta}(\vec{\phi})$ leaves $\vec{\phi}_0$ unchanged, that is $\vec{\Delta}(\vec{\phi}_0) = 0$, this means that our transformation is a symmetry transformation that is *not* broken, since it is respected by the ground state.[1] Then, M_{ji}^2 does not need to be zero to obey the condition above.

On the other hand, a spontaneously broken symmetry is precisely one for which its transformation has the property $\vec{\Delta}(\vec{\phi}_0) \neq 0$. In this case, $\vec{\Delta}(\vec{\phi}_0)$ is a vector with eigenvalue zero, thus with mass zero.

19.3 The Higgs Mechanism

19.3.1 From Complex to Real Scalar Field Components

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}^a\right)^2 + \left|D_\mu\phi\right|^2 - V(\phi),$$

where ϕ describes a vector, the components of which are the complex scalar fields ϕ_i. Let the potential V be such that this Lagrangian is invariant under a local gauge transformation

$$\phi_i(x) \to (1 + i\alpha^a(x)\,t^a)_{ij}\phi_j(x).$$

Obviously, the dimension of the matrices t^a must match the number of scalar fields ϕ_i.

It is convenient to write the ϕ_i as real-valued fields; we can always write n complex fields $\phi_1, \dots, \phi_n$ as $2n$ real fields $\phi_1, \dots, \phi_{2n}$. To see how this works in detail, we are now going to consider the relevant examples, namely SU(2) and U(1).

First, consider SU(2), that is $t^a = \sigma^a/2$ are 2×2 matrices and the vector has two complex components. Let us now express the two complex components by four real components ϕ_i as follows:

$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}-i\phi_1 - \phi_2\\ \phi_4 + i\phi_3\end{pmatrix}.$$

Admittingly, this definition of the four real components ϕ_i is quite awkward, but note that we simply have split the complex components ϕ_1, ϕ_2 into its real and imaginary part, absorbed minus signs into those unknown real and imaginary parts and arbitrarily numbered them in a weird order. It is just, that the results will appear most conveniently in this way.

The transformation rule given above for the complex fields now reads in terms of the new fields ϕ_i

$$-i\phi_1 - \phi_2 \to (1 + i\alpha^a t_{11}^a)(-i\phi_1 - \phi_2) + (1 + i\alpha^a t_{12}^a)(\phi_4 + i\phi_3),$$

$$\phi_4 + i\phi_3 \to (1 + i\alpha^a t_{21}^a)(-i\phi_1 - \phi_2) + (1 + i\alpha^a t_{22}^a)(\phi_4 + i\phi_3).$$

[1] For example, for $N = 3$ and the choice $\vec{\phi}_0 = (0, 0, v)$, there is a single rotation, the one around the z axis, which leaves $\vec{\phi}_0$ unchanged. This symmetry is *not* broken. However, the rotational symmetries around the x and y axes change our choice of $\vec{\phi}_0$, therefore those symmetries are broken.

The second Pauli matrix is imaginary, the first and third are real. Let us therefore introduce $\tilde{a} = 1, 3$ and write $\alpha^a t^a = \alpha^{\tilde{a}} t^{\tilde{a}} + \alpha^2 t^2$ (sum over $a = 1, 2, 3$ and $\tilde{a} = 1, 3$ implied). Then, the first term is purely real, the second purely imaginary. With this notation, the change of $-i\phi_1 - \phi_2$ (the difference of the transformed $-i\phi_1 - \phi_2$ and the old one) can be written as

$$\begin{aligned}\delta(-i\phi_1 - \phi_2) &= i\alpha^a t^a_{11}(-i\phi_1 - \phi_2) + i\alpha^a t^a_{12}(\phi_4 + i\phi_3) \\ &= \alpha^2 t^2_{11}\phi_1 - i\alpha^{\tilde{a}} t^{\tilde{a}}_{11}\phi_2 - \alpha^2 t^2_{12}\phi_3 + i\alpha^{\tilde{a}} t^{\tilde{a}}_{12}\phi_4 \\ &\qquad + \alpha^{\tilde{a}} t^{\tilde{a}}_{11}\phi_1 - i\alpha^2 t^2_{11}\phi_2 - \alpha^{\tilde{a}} t^{\tilde{a}}_{12}\phi_3 + i\alpha^2 t^2_{12}\phi_4\end{aligned}$$

We have arranged those eight terms in such a way, that the first four terms are imaginary and the second four terms are real. Since the ϕ_i are real, the first four terms need to be the correction to $-i\phi_1$ and the second four terms need to be the corrections to $-\phi_2$. Hence,

$$\delta\phi_1 = +i\alpha^2 t^2_{11}\phi_1 + \alpha^{\tilde{a}} t^{\tilde{a}}_{11}\phi_2 - i\alpha^2 t^2_{12}\phi_3 - \alpha^{\tilde{a}} t^{\tilde{a}}_{12}\phi_4 =: -\alpha^a T^a_{1j}\phi_j,$$

$$\delta\phi_2 = -\alpha^{\tilde{a}} t^{\tilde{a}}_{11}\phi_1 + i\alpha^2 t^2_{11}\phi_2 + \alpha^{\tilde{a}} t^{\tilde{a}}_{12}\phi_3 - i\alpha^2 t^2_{12}\phi_4 =: -\alpha^a T^a_{2j}\phi_j,$$

$$\delta\phi_3 = -i\alpha^2 t^2_{21}\phi_1 - \alpha^{\tilde{a}} t^{\tilde{a}}_{21}\phi_2 + i\alpha^2 t^2_{22}\phi_3 + \alpha^{\tilde{a}} t^{\tilde{a}}_{22}\phi_4 =: -\alpha^a T^a_{3j}\phi_j,$$

$$\delta\phi_4 = +\alpha^{\tilde{a}} t^{\tilde{a}}_{21}\phi_1 - i\alpha^2 t^2_{21}\phi_2 - \alpha^{\tilde{a}} t^{\tilde{a}}_{22}\phi_3 + i\alpha^2 t^2_{22}\phi_4 =: -\alpha^a T^a_{4j}\phi_j.$$

To get the formulas for $\delta\phi_{3,4}$, we could simply change the first index of the generator metrices from 1 to 2 and add a global minus sign. Also, we defined 4×4 matrices T^a_{ij} with $a = 1, 2, 3$ and $i, j = 1, 2, 3, 4$. We can read off their components from the equations above as follows, using $t^a = \sigma^a/2$:

$$T^1 = \begin{pmatrix} 0 & -t^1_{11} & 0 & t^1_{12} \\ t^1_{11} & 0 & -t^1_{12} & 0 \\ 0 & t^1_{21} & 0 & -t^1_{22} \\ -t^1_{21} & 0 & t^1_{22} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} -it^2_{11} & 0 & it^2_{12} & 0 \\ 0 & -it^2_{11} & 0 & it^2_{12} \\ it^2_{21} & 0 & -it^2_{22} & 0 \\ 0 & it^2_{21} & 0 & -it^2_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 0 & -t^3_{11} & 0 & t^3_{12} \\ t^3_{11} & 0 & -t^3_{12} & 0 \\ 0 & t^3_{21} & 0 & -t^3_{22} \\ -t^3_{21} & 0 & t^3_{22} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Note, that all those matrices T^a are real and antisymmetric. Since $\delta\phi_i = -\alpha^a T^a_{ij}\phi_j$, the transformation rules for the real components ϕ_i read

$$\phi_i = \left(1 - \alpha^a(x)\, T^a_{ij}\right)\phi_j, \qquad i, j = 1, 2, 3, 4.$$

Let us consider the other relevant example, U(1). In U(1) the complex components of ϕ transform as $\phi_i \to (1 + i\alpha(x)\, Y)_{ij}\phi_j(x)$, where Y is the U(1) charge (a real number). That is, $\delta\phi_i = i\alpha Y\delta_{ij} = i\alpha Y T_{ij}$ and we can redo the derivation above by replacing $t^a_{ij} \to Y\delta_{ij}$. Since Y is real, we should use a real matrix for that, that is, t^1 or t^3. No matter, which one of those we take, the corresponding matrix T is

$$T = \begin{pmatrix} 0 & -Y\delta_{11} & 0 & Y\delta_{12} \\ Y\delta_{11} & 0 & -Y\delta_{12} & 0 \\ 0 & Y\delta_{21} & 0 & -Y\delta_{22} \\ -Y\delta_{21} & 0 & Y\delta_{22} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -Y & 0 & 0 \\ Y & 0 & 0 & 0 \\ 0 & 0 & 0 & -Y \\ 0 & 0 & Y & 0 \end{pmatrix}$$

with the corresponding transformation

$$\phi_i = \left(1 - \alpha(x)\, T_{ij}\right)\phi_j.$$

Thus, effectively, we simply replace $t^a_{ij} \to iT^a_{ij}$ in the transformation rule; however, after the replacement the indices i, j take on the double number of values.

Also, in absolute square terms, we can write, according to our definition of the real components above,

$$\begin{aligned}|\phi|^2 = \phi^\dagger\phi = \phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 &\to \frac{1}{2}\Big((\phi_1 + i\phi_2)^\dagger(\phi_1 + i\phi_2) + (\phi_3 + i\phi_4)^\dagger(\phi_3 + i\phi_4)\Big) \\ &= \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^3 + \phi_4^2) = \frac{1}{2}\phi^2,\end{aligned}$$

where the ϕ after the last equal sign would be the real vector $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$. Combing these two results, we can replace

$$\left|D_\mu\phi\right|^2, \qquad \text{where} \qquad D_\mu = \partial_\mu + igA^a_\mu t^a, \qquad \phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{pmatrix} \in \mathbb{C}$$

by

$$\frac{1}{2}\left(D_\mu\phi\right)^2, \qquad \text{where} \qquad D_\mu = \partial_\mu - gA^a_\mu T^a, \qquad \phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbb{R}.$$

We use the same symbols ϕ for the vector and ϕ_i for its components, no matter whether they are real or complex. It will usually be clear from context, which one we mean (for example, if there is a T^a around, the field ϕ is meant to be scalar; if a t^a around, it's meant to be complex ϕ).

19.3.2 Expanding about the Vacuum Expectation Value

Using our considerations of expression a complex field vector ϕ by a real field vector ϕ of components ϕ_i by using the matrices T^a from (>19.3.1), we can write the kinetic term of the scalar field Lagrangian as

$$\begin{aligned}\left|D_\mu\phi\right|^2 &\to \frac{1}{2}\left(D_\mu\phi\right)^2 = \frac{1}{2}\left(\partial_\mu\phi - gA^a_\mu T^a\phi\right)^2 \\ &= \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}gA^\mu_b\left(\partial_\mu\phi\right)(T^b\phi) - \frac{1}{2}gA^a_\mu(T^a\phi)(\partial^\mu\phi) + \frac{1}{2}gA^a_\mu(T^a\phi)gA^\mu_b(T^b\phi) \\ &= \frac{1}{2}\left(\partial_\mu\phi\right)^2 - gA^\mu_b\left(\partial_\mu\phi_i\right)\left(T^b_{ij}\phi_j\right) + \frac{1}{2}g^2A^a_\mu A^\mu_b(T^a\phi)(T^b\phi).\end{aligned}$$

Note, that there is a dot product between the vectors $\partial_\mu\phi$ and $T^b\phi$ in the intermediate step.

We treated the fields of the linear sigma model from section 19.1 as classical fields; we found the minimum at $\phi = \phi_0$. Consider the classical limit $\hbar \to 0$ of the vacuum expectation value of a single field:

$$\lim_{\hbar\to 0}\,\langle\Omega|\phi|\Omega\rangle = \lim_{\hbar\to 0}\int \mathcal{D}\phi\;\phi\,\exp\left(\frac{i}{\hbar}\int d^4x\;\mathcal{L}[\phi]\right) = \phi_0.$$

Only the stationary value of ϕ, namely its value ϕ_0 where the Lagrangian has its minimum, dominates the path in integral. In the classical limit $\hbar \to 0$, we can even write the exponential as a δ-function $\delta(\phi_i - \phi_{0i})$. Thus, in the quantum case, it will at least give us the correct classical limit, if assume that the minima v of the potential are the vacuum expectation values. After all, the fact that the *vacuum* expectation value gives the *minimum* of the potential, does seem quite reasonable.

Let us therefore write the vacuum expectation values (i. e. the minima of the potential) as

$$\langle\Omega|\phi|\Omega\rangle = \phi_0.$$

Expanding around this minimum like $\phi(x) = \phi_0 + \phi'(x)$, we find for the kinetic terms of the fields ϕ_i

$$\begin{aligned}\frac{1}{2}\left(D_\mu\phi\right)^2 &= \frac{1}{2}\left(\partial_\mu\phi'\right)^2 - gA_b^\mu\left(\partial_\mu\phi_i'\right)\left(T_{ij}^b\left(\phi_{0j}+\phi_j'\right)\right) + \frac{1}{2}g^2A_\mu^aA_b^\mu\left(T^a(\phi_0+\phi')\right)\left(T^b(\phi_0+\phi')\right)\\ &= \frac{1}{2}\left(\partial_\mu\phi'\right)^2 - gA_a^\mu\left(\partial_\mu\phi_i'\right)\left(T_{ij}^a\phi_{0j}\right) + \frac{1}{2}\underbrace{g^2(T^a\phi_0)(T^b\phi_0)}_{=:m_{ab}^2}A_\mu^aA_b^\mu + \mathcal{O}((\text{fields})^3),\end{aligned}$$

where we omitted terms cubic and quartic in the fields ϕ_i', A_μ^a.

Note, that in the Klein-Gordon theory, the sign of the mass term is negative, whereas it seems to be positive of the gauge bosons. However, since $A_\mu^aA_b^\mu = A_0^aA_b^0 - A_i^aA_b^i$, the physical spacelike components indeed have an additional minus sign. Also, the masses m_{aa}^2 (the diagonal elements of m_{ab}^2) are positive:

$$m_{aa}^2 = g^2(T^a\phi_0)^2 \geq 0 \qquad (\text{no sum over } a),$$

since the T^a are real. In principle, however, they may be some particular generator, that leaves the vacuum value invariant, such that $T^a\phi_0 = 0$, and the gauge boson remains massless. Let us also define $F^a \coloneqq T^a\phi_0$ or, equivalently, $F_i^a \coloneqq T_{ij}^a\phi_{0j}$.

19.3.3 Only Goldstone Bosons contribute to the Vertex with the Gauge Boson

The second term of $\left(D_\mu\phi\right)^2/2$ describes a vertex between a single gauge boson line and a single scalar field line:

$$-gA_a^\mu\left(\partial_\mu\phi'\right)\underbrace{\left(T^b\phi_0\right)}_{=F^b} = -gA_a^\mu\partial_\mu(\phi'\cdot F^b).$$

We could put the derivative outside the parenthesis, since the vector F^a is constant. By the properties of the scalar product, only components of ϕ' parallel to the vector F^b will survive in this term. Since our transformation rule for the vacuum expectation value reads

$$\phi_0(x) \to (1-\alpha^a(x)T^a)\phi_0 = \phi_0 - \alpha^aF^a,$$

the direction of the vector F^a is the direction in which our transformation shifts ϕ_0. Recall, that in the linear sigma model, we chose $\phi_0 = (0,0,\dots,0,v)$ and the describes deviations from this point by $\phi = (\vec{\pi}, v+\sigma)$. For any rotation $R = 1 + \delta R$, we then have[1]

$$(\vec{\pi}, v+\sigma)\cdot(\delta R\,\phi_0) = (\vec{\pi}, v+\sigma)\cdot v\begin{pmatrix}\delta R_{1N}\\ \delta R_{2N}\\ \vdots\\ \delta R_{N-1,N}\\ 0\end{pmatrix}.$$

The components of $\delta R\,\phi_0$ which do not disappear correspond pricelessly to the components of $(\vec{\pi}, v+\sigma)$ which contain the Goldstone bosons $\vec{\pi}$; those are the only components of the vector

---

[1] A general infinitesimal three dimensional rotation about an infinitesimal angle θ and an axis $\vec{n}$ is given by the matrix

$$1+\delta R = 1 + \begin{pmatrix}0 & -\theta\,n_z & \theta\,n_y\\ \theta\,n_z & 0 & -\theta\,n_x\\ -\theta\,n_y & \theta\,n_x & 0\end{pmatrix}.$$

Thus, for the vector $\phi_0 = (0,0,v)$, we find

$$\delta R\,\phi_0 = v\begin{pmatrix}\theta\,n_y\\ -\theta\,n_x\\ 0\end{pmatrix},$$

with no z-component.

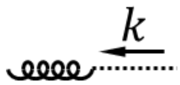
$(\vec{\pi}, v+\sigma)$ which contribute. By analogy, the components of ϕ' that contribute to $\phi' \cdot F^b$ are precisely the Goldstone bosons.

19.3.4 The Vertex Factor of the Gauge Boson Goldstone Boson Vertex

Recall, that Goldstone bosons are massless. Thus, only the massless components of ϕ' contribute to the vertex described by

$$-gA_a^\mu(\partial_\mu \phi')F^b = -gA_a^\mu(\partial_\mu \phi_i')F_i^b.$$

It can be drawn as a Feynman diagram like



We can derive the Feynman rules from this interaction term as in (>18.1.4) by considering the Matrix element of the term with other fields that can be around:

$$-igF_i^b\, \partial_\mu \left\langle \Omega \middle| \left(\cdots A_c^\nu(x_1)\phi_k'(x_2)\right)\left(A_a^\mu(z)\, \partial_\mu \phi_i'(z)\right) \middle| \Omega \right\rangle = -igF_i^b\, \widehat{D}_{F,ac}^{\mu\nu}(x_1 - z)\, \partial_{z\mu} D_{F,ik}(z - x_2).$$

Here, $D_{F,ij}$ is the propagator of the scalar field from section 4.8 (times δ_{ij}). It obeys $D_{F,ij}(x_2 - z) = D_{F,ij}(z - x_2)$ and since we have chosen the momentum k to point *into* the vertex, we should choose the latter Feynman propagator, describing a particle propagating from x_2 to z. Then, using the explicit formula from section 4.8, we can perform the derivative $\partial_{z\mu}$ (that is a derivative with respect to z_μ) and get

$$-igF_i^b\left(-ik_\mu\right) \widehat{D}_{F,ac}^{\mu\nu}(x_1 - z)\, \partial_{z\mu} D_{F,ik}(z - x_2) = -gk_\mu F_i^b\, \widehat{D}_{F,ac}^{\mu\nu}(x_1 - z)\, \partial_{z\mu} D_{F,ik}(z - x_2).$$

Thus, the Feynman rule for the gauge boson Goldstone boson vertex is

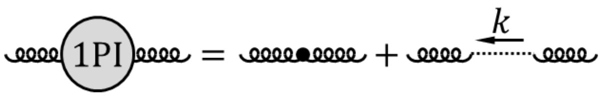
$$-gk^\mu F^a.$$

19.3.5 1PI of the Gauge Boson

If we also treat the mass term

$$\frac{1}{2}\left(D_\mu \phi\right)^2 = \cdots + \frac{m_{ab}^2}{2} A_\mu^a A_b^\mu + \mathcal{O}((\text{fields})^3)$$

as a perturbation, the leading-order contributions to the 1PI of the gauge boson propagator read



or, in terms of mathematics,

$$1\text{PI} = im_{ab}^2\eta^{\mu\nu} + (-gk^\mu F^a)\frac{i}{k^2}(gk^\nu F^b) = im_{ab}^2\eta^{\mu\nu} - i\,\underbrace{g^2F^aF^b}_{=m_{ab}^2}\frac{k^\mu k^\nu}{k^2} = im_{ab}^2\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right).$$

Note, that this is an exact equality, since the 1PI contains only these two diagrams arising from interactions with Goldstone bosons only.

19.4 The Glashow-Weinberg-Salam Theory of Weak Interactions

19.4.1 SU(2) Invariance yields three massive Gauge Bosons

The starting point of GWS theory is a SU(2) gauge field coupled to scalar fields ϕ_i. Let us use the fundamental representation of the t's, where $t^a = \sigma^a/2$ with the Pauli matrices σ^a. Since the Pauli matrices are two dimensional, we have two scalar fields ϕ_1, ϕ_2.

Let the (complex) field $\phi = (\phi_1, \phi_2)$ acquire a vacuum expectation value (which minimizes the Lagrangian) of the form

$$\phi_0 = \frac{1}{\sqrt{2}}\binom{0}{v}.$$

Expanding $\phi = \phi_0 + \phi'$ and plugging this expansion into the squared covariant derivative $D_\mu\phi = (\partial_\mu + igA_\mu^a t^a)\phi$, the term which we can read of the mass from reads

$$\begin{aligned}|D_\mu\phi|^2 &= |(\partial_\mu + igA_\mu^a t^a)\phi|^2 = \left((\partial_\mu + igA_\mu^a t^a)\phi\right)^\dagger (\partial^\mu + igA_b^\mu t^b)\phi \\ &= \cdots - \phi^\dagger(igA_\mu^a t^a)(igA_b^\mu t^b)\phi = \cdots + g^2\phi_0^\dagger t^a t^b \phi_0 A_\mu^a A_b^\mu + \cdots = \cdots + \frac{1}{4}g^2\, \phi_0^\dagger\phi_0\, A_\mu^a A_a^\mu + \cdots.\end{aligned}$$

Here, we neglected terms with ϕ', since those are not mass terms for the gauge boson. In the last step, we used

$$t^a t^b\, A_\mu^a A_b^\mu = \frac{1}{2}\{t^a, t^b\}\, A_\mu^a A_b^\mu = \frac{1}{8}\underbrace{\{\sigma^a, \sigma^b\}}_{=2\delta^{ab}}\, A_\mu^a A_b^\mu = \frac{1}{4}\delta^{ab}\, A_\mu^a A_b^\mu.$$

Thus, the gauge boson fields A_μ^a have received a mass

$$m_A^2 := \frac{1}{2}g^2\phi_0^\dagger\phi_0 = \frac{1}{4}g^2 v^2.$$

SU(2) has $2^2 - 1 = 3$ generators, thus there is a gauge boson for each index $a = 1, 2, 3$. All of them have the same mass m_A.[1]

19.4.2 Including the massless Photon – Kinetic Term of the Scalar Field (Explicit Derivation)

To include the massless photon into the formalism, we need to construct a Lagrangian, which Is additionally invariant under a U(1) symmetry, that is under a transformation

$$\phi \to e^{i\alpha^a t^a} e^{i\beta/2}\phi,$$

where, again, $t^a = \sigma^a/2$. The respective covariant derivative will be

[1] This does not have to be the case. It was due to the property $t^a t^b \sim \delta^{ab}$, which is a specific to the representation of t^a as the Pauli matrices. Other representations may well have the property that, for example, $t^3 t^3 = 0$. Then, the third gauge boson stay massless. Indeed, this is the case for real fields ϕ_i in the adjoint representation, where $(t^a)_{bc} = -if^{abc}$ and hence

$$\begin{aligned}(D_\mu\phi)_a &= \partial_\mu\phi_a + igA_\mu^b (t^b)_{ac}\phi_c = \partial_\mu\phi_a + gA_\mu^b f^{bac}\phi_c \\ \Leftrightarrow \qquad \frac{1}{2}(D_\mu\phi)^2 &= \cdots + \frac{1}{2}(gA_\mu^b f^{bac}\phi_c)(gA_d^\mu f^{dae}\phi_e) = \cdots + \frac{1}{2}g^2 f^{ade} f^{abc}\phi_c\phi_e A_\mu^b A_d^\mu.\end{aligned}$$

Assuming $\phi_0 = (0, 0, v)$, that is $\phi_a = v\,\delta_{3c}$ yields

$$\frac{1}{2}(D_\mu\phi)^2 = \cdots + \frac{1}{2}g^2 v^2 f^{ad3} f^{ab3} A_\mu^b A_d^\mu.$$

Recall from section 2.2, that f^{abc} is totally antisymmetric (in fact, since in this case the indices take three values, f^{abc} is precisely the Levi-Civita symbol). Hence, there is obviously *no* mass term $\sim A_\mu^3 A_3^\mu$ and the third gauge boson remains massless. In this way, we could describe two massive W bosons and a massless photon and thereby unify weak and electromagnetic interactions. Indeed, this model was a serious candidate for that purpose; however, nature chooses to have a Z boson as well.

$$D_\mu = \partial_\mu + igA^a_\mu t^a + \frac{ig'}{2}B_\mu.$$

Using this covariant derivative, we find

$$\begin{aligned}
|D_\mu\phi|^2 &= \left|\left(\partial_\mu + igA^a_\mu t^a + ig'B_\mu/2\right)\phi\right|^2 \\
&= \left(\left(\partial_\mu + igA^a_\mu t^a + ig'B_\mu/2\right)\phi\right)^\dagger \left(\partial^\mu + igA^\mu_b t^b + ig'B^\mu/2\right)\phi \\
&= \phi^\dagger\left(\overleftarrow{\partial}_\mu + igA^a_\mu t^a + ig'B_\mu/2\right)^\dagger\left(\partial^\mu + igA^\mu_b t^b + ig'B^\mu/2\right)\phi \\
&= \phi^\dagger\left(\overleftarrow{\partial}_\mu - igA^a_\mu t^a - ig'B_\mu/2\right)\left(\partial^\mu + igA^\mu_b t^b + ig'B^\mu/2\right)\phi \\
&= \phi^\dagger\left(\overleftarrow{\partial}_\mu\partial^\mu\right)\phi + \phi^\dagger\left(\overleftarrow{\partial}_\mu igA^\mu_b t^b + \overleftarrow{\partial}_\mu\, ig'B^\mu/2 - igA^a_\mu t^a\partial^\mu - ig'B_\mu/2\,\partial^\mu\right)\phi \\
&\quad + \phi^\dagger\left(g^2 t^a t^b A^a_\mu A^\mu_b + gg' t^a A^a_\mu B^\mu + g'^2 B_\mu B^\mu/4\right)\phi
\end{aligned}$$

We have split the last expression into three parts. As explained in section 18.3, the latter two of them will equip the gauge bosons with a mass; when we plug in $\phi(x) = \phi_0 + \phi'(x)$ in the last term, the mass arises directly from the term $\phi_0^\dagger$(third term)ϕ_0. The second term yields a vertex between ϕ' and the gauge bosons, that finally also yield a contribution to the mass term, by the mechanism of section 18.3.

Since the second term is "only" needed to ensure the right form of the propagator of massive gauge bosons, it is sufficient to consider the last term only to find the masses of the gauge boson. Let us therefore consider this last term only.

The last term itself contains another three terms inside its bracket. Consider those three terms within the last terms individually, after plugging in $\phi(x) = \phi_0 + \phi'(x)$ and considering only terms without any appearance of ϕ' (note, that the first one is the same as in (>19.4.1)):

$$\phi_0^\dagger\left(g^2 t^a t^b A^a_\mu A^\mu_b\right)\phi_0 = \frac{1}{8}g^2 v^2 A^a_\mu A^\mu_a,$$

$$\phi_0^\dagger\left(gg' t^a A^a_\mu B^\mu\right)\phi_0 = \frac{1}{4}gg'v^2\,(0,1)(\sigma^a)\begin{pmatrix}0\\1\end{pmatrix}A^a_\mu B^\mu = \frac{1}{4}gg'v^2\,\sigma^a_{22}A^a_\mu B^\mu = -\frac{1}{4}gg'v^2\,A^3_\mu B^\mu,$$

$$\phi_0^\dagger\left(g'^2 B_\mu B^\mu/4\right)\phi_0 = \frac{1}{8}g'^2 v^2 B^2.$$

We can give the combined result in the following form:

$$|D_\mu\phi|^2 = \cdots + \frac{1}{8}v^2\left(g^2\left(A^1_\mu\right)^2 + g^2\left(A^2_\mu\right)^2 + \left(-gA^3_\mu + g'B_\mu\right)^2\right) + \cdots = \frac{m^2_{ab}}{2}A^a_\mu A^\mu_b + \cdots,$$

where

$$m^2_{ab} = \frac{v^2}{4}\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix}^{ab}, \qquad A^4_\mu = B_\mu.$$

19.4.3 Including the massless Photon – Kinetic Term of the Scalar Field (Alternative Derivation)

In principle, we have done the calculation from (>19.4.2) already in a more general way in section 18.3. What we found was, that the mass term reads

$$\frac{m^2_{ab}}{2}A^a_\mu A^b_\mu, \qquad m^2_{ab} = g^2 F^a_i F^b_i, \qquad F^a_i \coloneqq T^a_{ij}\phi_{0j}.$$

We computed the matrices T^a in (>19.3.1) for the SU(2) and U(1) case. In GWS theory, we have the combined symmetries SU(2) × U(1). We can account for that fact by simply letting the SU(2) indices

$a = 1, 2, 3$ take an additional value 4. Then, t_{ij}^4 will be the U(1) generator $Y\delta_{ij}$. The corresponding real "generator" T_{ij}^4 is then precisely the matrix that we simply called T in (>19.3.1). In GWS theory, we use the U(1) charge $Y = 1/2$. Also, the U(1) coupling constant g' differs from the SU(2) coupling g. We take care of that by writing

$$m_{ab}^2 = g^a g^b F_i^a F_i^b \quad \text{(sum over } i\text{, but no sum over } a, b\text{)}, \qquad a, b, i = 1, 2, 3, 4.$$

When the complex scalar field reads $\phi_0 = (0, v)/\sqrt{2}$, its real counter part reads $\phi_0 = (0, 0, 0, v)$, according to (>19.3.1), that is $\phi_{0j} = v\delta_{i4}$. Then, using the matrices T^a with $T^4 = T$ from (>19.3.1),

$$(gF)^{ai} := g^a F_i^a = g^a T_{ij}^a \phi_{0j} = v g^a T_{i4}^a = \frac{v}{2}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & -g' & 0 \end{pmatrix}^{ai} \qquad \text{(no sum over } a\text{)}$$

Note that the row index is a and the column index is i. It is kind of a coincident, that the matrix $g^a F_i^a$ is square, since a and i only take on the same number of values in the special case of GWS theory.

Then, we can write the mass matrix as

$$\begin{aligned} m_{ab}^2 &= (g^a F_i^a)(g^b F_i^b) = (gF)^{ai}(gF)^{bi} = \left((gF)(gF)^T\right)^{ab} \\ &= \frac{v^2}{4}\left(\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & -g' & 0 \end{pmatrix}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & g & -g' \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)^{ab} = \frac{v^2}{4}\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix}^{ab}. \end{aligned}$$

Obviously, this exactly corresponds to the final result from (>19.4.2).

19.4.4 Including the massless Photon – The A and the Z Field

To express the Lagrangian in terms of mass eigenstates, we want to diagonalize the mass matrix m_{ab}^2. Obviously, the upper left quarter of this matrix is already diagonalized. Thus, we only need to diagonalize the lower right quarter:

$$M := \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{cc} \text{Eigenvalues:} & \text{Eigenvectors:} \\ 0 & (g', g) \\ g^2 + g'^2 & (g, -g') \end{array} \quad \Rightarrow \quad S := \frac{1}{\sqrt{g^2 + g'^2}}\begin{pmatrix} g' & g \\ g & -g' \end{pmatrix}.$$

Thus, $D = S^{-1}MS$ is the diagonalized matrix to M and we can write

$$(A_\mu^3, B_\mu)\, M \begin{pmatrix} A_3^\mu \\ B^\mu \end{pmatrix} = (A_\mu^3, B_\mu)\, SS^{-1}MSS^{-1}\begin{pmatrix} A_3^\mu \\ B^\mu \end{pmatrix} = (A_\mu^3, B_\mu)\, SDS^{-1}\begin{pmatrix} A_3^\mu \\ B^\mu \end{pmatrix}$$

$$(A_\mu^3, B_\mu)\, S = \frac{1}{\sqrt{g^2 + g'^2}}(g' A_\mu^3 + g B_\mu, g A_\mu^3 - g' B_\mu).$$

Let us abbreviate

$$A_\mu := -\frac{1}{\sqrt{g^2 + g'^2}}(g' A_\mu^3 + g B_\mu), \qquad Z_\mu^0 := -\frac{1}{\sqrt{g^2 + g'^2}}(g A_\mu^3 - g' B_\mu)$$

and we find

$$|D_\mu \phi|^2 = \cdots + \frac{1}{8}v^2\left(g^2 (A_\mu^1)^2 + g^2 (A_\mu^2)^2 + (A_\mu, Z_\mu^0)\begin{pmatrix} 0 & 0 \\ 0 & g^2 + g'^2 \end{pmatrix}\begin{pmatrix} A_\mu \\ Z_\mu^0 \end{pmatrix}\right)\cdots.$$

Now it is obvious, that A_μ is massless and Z_μ^0 has the mass $m_Z^2 = v^2(g^2 + g'^2)/4$. For future reference, let us also invert the definitions of A_μ, Z_μ^0:

$$A_\mu^3 = -\frac{1}{\sqrt{g^2+g'^2}}\left(g' A_\mu + g Z_\mu^0\right), \qquad B_\mu = -\frac{1}{\sqrt{g^2+g'^2}}\left(g A_\mu - g' Z_\mu^0\right).$$

Since

$$\left(\frac{g}{\sqrt{g^2+g'^2}}\right)^2 + \left(\frac{g'}{\sqrt{g^2+g'^2}}\right)^2 = 1$$

it is convenient to define the *weak mixing angle* by

$$\cos\theta_w \coloneqq \frac{-g}{\sqrt{g^2+g'^2}} \qquad \Longrightarrow \qquad \sin\theta_w = \frac{-g'}{\sqrt{g^2+g'^2}}.$$

This definition allows us to write

$$\begin{pmatrix} Z_\mu^0 \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix} \qquad \Leftrightarrow \qquad \begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} Z_\mu^0 \\ A_\mu \end{pmatrix}.$$

19.4.5 Including the massless Photon – The W Fields

We define

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \mp i A_\mu^2\right)$$

$$\Leftrightarrow \qquad A_\mu^1 = \frac{1}{\sqrt{2}}\left(W_\mu^- + W_\mu^+\right), \qquad A_\mu^2 = \frac{1}{\sqrt{2}i}\left(W_\mu^- - W_\mu^+\right).$$

Hence, the mass terms in the Lagrangian take on the form

$$\begin{aligned} \left|D_\mu \phi\right|^2 &= \cdots + \frac{1}{8} v^2 \left(g^2 \left(\left(A_\mu^1\right)^2 + \left(A_\mu^2\right)^2\right) + (g^2 + g'^2)\left(Z_\mu^0\right)^2\right) \\ &= \cdots + \frac{1}{4} g^2 v^2\, W_\mu^- W^{+\mu} + \frac{1}{8}(g^2+g'^2) v^2 \left(Z_\mu^0\right)^2 = \cdots + m_W^2\, W_\mu^- W^{+\mu} + \frac{m_Z^2}{2}\left(Z_\mu^0\right)^2 \end{aligned}$$

Recall from section 4.7, that *complex* boson fields have a mass term $-m^2\phi^*\phi$ in the Lagrangian. Note also, that $W^- = (W^+)^*$. Thus, we needed to introduce the fields $W_\mu^\pm$ in favour of $A_\mu^{1,2}$ to get the right mass term (also the sign for the physical, spatial components is correct: $W_\mu^- W^{+\mu} = W_0^- W^{+0} - W_i^- W^{+i}$; same for the field Z_μ^0).

19.4.6 The Covariant Derivative in Terms of the New Fields

Let's find the covariant derivative

$$D_\mu = \partial_\mu + i g A_\mu^a t^a + i g' Y B_\mu$$

in terms of the new fields $W_\mu^\pm, Z_\mu^0$ and A_μ. Introducing

$$t^\pm \coloneqq t^1 \pm i t^2 \qquad \Leftrightarrow \qquad t^+ + t^- = 2t^1, \qquad t^+ - t^- = 2it^2$$

and plugging in the formulas for $A_\mu^{1,2}$ in terms of the fields $W_\mu^\pm$ from (>19.4.5), the first two terms of the sum over a look as follows:

$$\begin{aligned} i g A_\mu^1 t^1 + i g A_\mu^2 t^2 &= \frac{ig}{2\sqrt{2}}\left(W_\mu^- + W_\mu^+\right)(t^+ + t^-) + \frac{ig}{2i\sqrt{2}i}\left(W_\mu^- - W_\mu^+\right)(t^+ - t^-) \\ &= \frac{ig}{\sqrt{2}}\left(W_\mu^+ t^+ + W_\mu^- t^-\right). \end{aligned}$$

By the formulas for A^3_μ and B_μ in terms of A_μ and Z^0_μ from (>19.4.4) we find for the remaining terms

$$\begin{aligned} igA^3_\mu t^3 + ig'YB_\mu &= ig\big(\cos\theta_w Z^0_\mu + \sin\theta_w A_\mu\big)t^3 + ig'Y\big(-\sin\theta_w Z^0_\mu + \cos\theta_w A_\mu\big) \\ &= i(g\cos\theta_w t^3 - g'\sin\theta_w Y)Z^0_\mu + i(g\sin\theta_w t^3 + g'\cos\theta_w Y)A_\mu \\ &= i(g\cos\theta_w t^3 - g'\sin\theta_w Y)Z^0_\mu + i\underbrace{g\sin\theta_w}_{=-e}\underbrace{(t^3+Y)}_{=Q}A_\mu. \end{aligned}$$

In the last step we used that, by the definition of θ_w, it holds $g\sin\theta_w = g'\cos\theta_w$. Let us now identify the electric charge $e = -g\sin\theta_w > 0$ and the charge number of the electron $Q = t^3 + Y = -1$, such that the coupling of the electron to the electromagnetic field falls into its conventional form.

The prefactor of Z^0_μ can be brought into the form

$$\begin{aligned} g\cos\theta_w t^3 - g'\sin\theta_w Y &= \frac{g^2t^3 - g'^2Y}{\sqrt{g^2+g'^2}} = \frac{(g^2+g'^2)t^3 - g'^2(Y+t_3)}{\sqrt{g^2+g'^2}} \\ &= \sqrt{g^2+g'^2}t^3 + g'\sin\theta_w Q = -\frac{g}{\cos\theta_w}t^3 + g\frac{\sqrt{g^2+g'^2}}{g}\frac{g'}{\sqrt{g^2+g'^2}}\sin\theta_w Q \\ &= -\frac{g}{\cos\theta_w}(t^3 - \sin^2\theta_w Q). \end{aligned}$$

Thus, all terms together, read

$$D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}}\big(W^+_\mu t^+ + W^-_\mu t^-\big) - \frac{ig}{\cos\theta_w}(t^3 - \sin^2\theta_w Q)Z^0_\mu - ieQA_\mu,$$

where $Q = -1$ for the electron.

19.5 Coupling to Fermions

19.5.1 Left-Handed Electrons and Neutrinos

We can describe left-handed electrons/positrons e_L and neutrinos ν_L together as a spinor

$$\psi \to E_L := \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}.$$

Doing so, we want to find $\bar{E}_L i\not{D}E_L$. Using the form of the covariant derivative in terms of the new fields, we find

$$\begin{aligned} \bar{E}_L i\not{D}E_L &= \bar{E}_L i\gamma^\mu\big(\partial_\mu + igA^a_\mu t^a + ig'YB_\mu\big)E_L \\ &= \bar{E}_L i\gamma^\mu\left(\partial_\mu + \frac{ig}{\sqrt{2}}\big(W^+_\mu t^+ + W^-_\mu t^-\big) - \frac{ig}{\cos\theta_w}(t^3 - \sin^2\theta_w Q)Z^0_\mu - ieQA_\mu\right)E_L. \end{aligned}$$

The term with the $W^\pm_\mu$ can be given as

$$\bar{E}_L i\gamma^\mu\left(\frac{ig}{\sqrt{2}}\big(W^+_\mu t^+ + W^-_\mu t^-\big)\right)E_L = gW^+_\mu\underbrace{\left(-\frac{1}{\sqrt{2}}\bar{E}_L\gamma^\mu t^+E_L\right)}_{=:J^{+\mu}_W} + gW^-_\mu\underbrace{\left(-\frac{1}{\sqrt{2}}\bar{E}_L\gamma^\mu t^-E_L\right)}_{=:J^{-\mu}_W},$$

where[1]

[1] When we wrote down the Feynman rules for QED in section 8.2, we only considered real polarization vectors. In general, they can be complex and the outgoing photon will receive a complex conjugate polarization vector ε^*_μ instead of ε_μ. So will the $W^\pm$ boson. Consider the vertex $\sim W^+_\mu J^{+\mu}_W \sim \bar{\nu}_L\not{W}^+e_L$. It can connect an incoming W^+ or an outgoing W^- to a fermion line.

$$J_W^{+\mu} = -\frac{1}{\sqrt{2}}(\bar{\nu}_L, \bar{e}_L)\gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} = -\frac{1}{\sqrt{2}}\bar{\nu}_L\gamma^\mu e_L,$$

$$J_W^{-\mu} = -\frac{1}{\sqrt{2}}(\bar{\nu}_L, \bar{e}_L)\gamma^\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \nu_{eL} \\ e_L^- \end{pmatrix} = -\frac{1}{\sqrt{2}}\bar{e}_L\gamma^\mu \nu_L.$$

The Z_μ^0 term yields

$$\bar{E}_L i\gamma^\mu \left(-\frac{ig}{\cos\theta_w}(t^3 - \sin^2\theta_w\, Q)Z_\mu^0\right)E_L = gZ_\mu^0 \underbrace{\left(\frac{1}{\cos\theta_w}\bar{E}_L\gamma^\mu(t^3 - \sin^2\theta_w\, Q)E_L\right)}_{=:J_Z^\mu}.$$

If we use $Y = -1/2$, we find

$$Q = t^3 + Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{2} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

which gives the right charge numbers for the neutrino and the electron. Thus, we can write the current as

$$\begin{aligned} J_Z^\mu &= \frac{1}{\cos\theta_w}(\bar{\nu}_L, \bar{e}_L)\gamma^\mu \left(\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \sin^2\theta_w \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\right)\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ &= \frac{1}{\cos\theta_w}\left(\frac{1}{2}\bar{\nu}_L\gamma^\mu\nu_L + \bar{e}_L\gamma^\mu\left(-\frac{1}{2} + \sin^2\theta_w\right)e_L\right). \end{aligned}$$

Finally,

$$\bar{E}_L i\gamma^\mu(-ieQA_\mu)E_L = eA_\mu \underbrace{(\bar{E}_L\gamma^\mu Q E_L)}_{=:J_{EM}^\mu}, \qquad J_{EM}^\mu = (\bar{\nu}_L, \bar{e}_L)\gamma^\mu \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = -\bar{e}_L\gamma^\mu e_L.$$

Using these definitions for the different currents, we can write

$$\bar{E}_L i\not{D}E_L = \bar{E}_L i\not{\partial}E_L + g\left(W_\mu^+ J_W^{+\mu} + W_\mu^- J_W^{-\mu} + Z_\mu^0 J_Z^\mu\right) + eA_\mu J_{EM}^\mu.$$

19.5.2 Right-Handed Electrons

Since the right-handed electrons are not allowed to interact with the $W_\mu^\pm$ fields, we simply set $t^a = 0$ for them. Thereby, they become singlets under the SU(2) transformation: $\psi \to e_R$. We then want $Q = t^3 + Y = Y = -1$. This yields

$$\begin{aligned} \bar{e}_R i\not{D}e_R &= \bar{e}_R i\not{\partial}e_R + \bar{e}_R i\gamma^\mu\left(\frac{ig}{\cos\theta_w}\sin^2\theta_w\, QZ_\mu^0 - ieQA_\mu\right)e_R \\ &= \bar{e}_R i\not{\partial}e_R + gZ_\mu^0 \underbrace{\left(\bar{e}_R\gamma^\mu\frac{\sin^2\theta_w}{\cos\theta_w}e_R\right)}_{=:J_Z^\mu} + eA_\mu \underbrace{(-\bar{e}_R\gamma^\mu e_R)}_{=:J_{EM}^\mu}. \end{aligned}$$

It is conventional to combine the electromagnetic currents from the right- and left-handed electron:

$$J_{EM,\text{tot}}^\mu = -\bar{e}_R\gamma^\mu e_R - \bar{e}_L\gamma^\mu e_L =: -\bar{e}\gamma^\mu e.$$

Note, that there are no right-handed neutrinos at all (in the standard model).

19.5.3 Left-Handed Quarks

Also different left-handed species of quarks can be written as a doublet. Consider for example an up- and down-quark doublet

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}.$$

This does not change anything for the $W_\mu^\pm$ couplings; with exactly the same derivation as in (>19.5.1), we find

$$\bar{q}_L i\not{D} q_L = \cdots + \; g W_\mu^+ J_W^{+\mu} + g W_\mu^- J_W^{-\mu} + \cdots,$$

where

$$J_W^{+\mu} = -\frac{1}{\sqrt{2}} \bar{u}_L \gamma^\mu d_L, \qquad J_W^{-\mu} = -\frac{1}{\sqrt{2}} \bar{d}_L \gamma^\mu u_L.$$

If we use $Y = 1/6$, we find

$$Q = t^3 + Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{6} = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix},$$

which gives us the right quark charges. Using this Q, the Z_μ^0 term yields

$$\begin{aligned} \bar{q}_L i\not{D} q_L &= \cdots + \bar{q}_L i\gamma^\mu \left(-\frac{ig}{\cos\theta_w} (t^3 - \sin^2\theta_w\, Q) Z_\mu^0 \right) q_L + \cdots \\ &= \cdots + g Z_\mu^0 \underbrace{\left(\bar{q}_L \gamma^\mu \frac{1}{\cos\theta_w} (t^3 - \sin^2\theta_w\, Q) q_L \right)}_{=J_Z^\mu} + \cdots, \end{aligned}$$

where

$$\begin{aligned} J_Z^\mu &= \frac{1}{\cos\theta_w} (\bar{u}_L, \bar{d}_L) \gamma^\mu \left(\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \sin^2\theta_w \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} \right) \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\ &= \frac{1}{\cos\theta_w} \left(\bar{u}_L \gamma^\mu \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w \right) u_L + \bar{d}_L \gamma^\mu \left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_w \right) d_L \right). \end{aligned}$$

Finally,

$$\bar{q}_L i\not{D} q_L = \cdots + \bar{q}_L i\gamma^\mu (-ieQA_\mu) q_L = \cdots + eA_\mu \underbrace{(\bar{q}_L \gamma^\mu Q q_L)}_{=:J_{EM}^\mu},$$

where

$$J_{EM}^\mu = (\bar{u}_L, \bar{d}_L) \gamma^\mu \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \frac{2}{3} \bar{u}_L \gamma^\mu u_L - \frac{1}{3} \bar{d}_L \gamma^\mu d_L.$$

19.5.4 Right-Handed Quarks

Right-handed quarks can be treated in the same way as the right-handed electrons from (>19.5.2). Simply set $t^a = 0$ and $Q = Y = 2/3$ for the u_R quark and $Q = Y = -1/3$ for the d_R quark. Then,

$$\begin{aligned} \bar{u}_R i\not{D} u_R &= \bar{u}_R i\not{\partial} u_R + \bar{u}_R i\gamma^\mu \left(\frac{ig}{\cos\theta_w} \sin^2\theta_w\, Q Z_\mu^0 - ieQA_\mu \right) u_R \\ &= \bar{u}_R i\not{\partial} u_R + g Z_\mu^0 \underbrace{\left(-\frac{2}{3} \bar{u}_R \gamma^\mu \frac{\sin^2\theta_w}{\cos\theta_w} u_R \right)}_{=:J_Z^\mu} + eA_\mu \underbrace{\left(\frac{2}{3} \bar{u}_R \gamma^\mu u_R \right)}_{=:J_{EM}^\mu}, \end{aligned}$$

$$\begin{aligned} \bar{d}_R i\not{D} d_R &= \bar{d}_R i\not{\partial} d_R + \bar{d}_R i\gamma^\mu \left(\frac{ig}{\cos\theta_w} \sin^2\theta_w\, Q Z_\mu^0 - ieQA_\mu \right) d_R \\ &= \bar{d}_R i\not{\partial} d_R + g Z_\mu^0 \underbrace{\left(\frac{1}{3} \bar{d}_R \gamma^\mu \frac{\sin^2\theta_w}{\cos\theta_w} d_R \right)}_{=:J_Z^\mu} + eA_\mu \underbrace{\left(-\frac{1}{3} \bar{d}_R \gamma^\mu d_R \right)}_{=:J_{EM}^\mu}. \end{aligned}$$

Again, it is convenient to combine the electromagnetic currents from the right- and left-handed quarks as follows:

up-quark: $\quad J^{\mu}_{EM,\text{tot}} = \frac{2}{3}\bar{u}_R\gamma^\mu u_R + \frac{2}{3}\bar{u}_L\gamma^\mu u_L = \frac{2}{3}\bar{u}\gamma^\mu u,$

down-quark: $\quad J^{\mu}_{EM,\text{tot}} = -\frac{1}{3}\bar{d}_R\gamma^\mu d_R - \frac{1}{3}\bar{d}_L\gamma^\mu d_L = -\frac{1}{3}\bar{d}\gamma^\mu d.$

19.6 Fermion Mass Terms

19.6.1 Notes regarding the fermion mass term (no reference in overview)

In QED, the fermion mass term is given by $-m\bar{\psi}\psi$. Using that γ^5 is Hermitian, we find that so are the projectors $P_{R,L} = (1 \pm \gamma^5)/2$. Also, since γ^5 anticommutes with any γ^μ, we find $P_{R,L}\gamma^\mu = \gamma^\mu P_{L,R}$. Thus,

$$\bar{\psi}_{R,L}\psi_{R,L} = \left(P_{R,L}\psi\right)^\dagger \gamma^0 P_{R,L}\psi = \psi^\dagger P_{R,L}\gamma^0 P_{R,L}\psi = \psi^\dagger \gamma^0 \underbrace{P_{L,R}P_{R,L}}_{=0}\psi = 0.$$

From this we find that

$$\bar{\psi}\psi = (\bar{\psi}_R + \bar{\psi}_L)(\psi_R + \psi_L) = \underbrace{\bar{\psi}_R\psi_R}_{=0} + \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R + \underbrace{\bar{\psi}_L\psi_L}_{=0} = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R.$$

19.6.2 The need for the Scalar Field Coupling in Mass Terms

We know from 19.5, that for the left-handed neutrino-electron spinor $\mathcal{E}_L = (\nu_L, e_L)$, the U(1) charge is $Y_L = -1/2$ and for the right-handed electron e_R it is $Y_R = -1$. Also, for e_R we used $t_R^a = 0$ in contrast to E_L, where $t_L^a = \sigma^a/2$. Thus, a mass term of the form $-m(\bar{e}_L e_R + \text{h.c.})$ would not be gauge invariant under a $\text{U}(1) \times \text{SU}(2)$ transformation:

$$-m\,\bar{e}_L e_R \to -m\,\bar{e}_L\, e^{-i\alpha^a t_L^a} e^{-i\beta Y_L} e^{i\beta Y_R} e^{-i\alpha^a t_R^a}\, e_R \neq -m\,\bar{e}_L e_R,$$

since $t_L^a \neq t_R^a$ and $Y_L \neq Y_R$.

A solution to the problem will be to include the Higgs field into the mass term; consider the term $-\lambda_e(\bar{\mathcal{E}}_L \cdot \phi)e_R + \text{h.c.}$ The scalar field has, according to section 18.4, $Y = 1/2$ and $t^a = \sigma^a/2 = t_L^a$. Then,

$$-\lambda_e(\bar{\mathcal{E}}_L \cdot \phi)e_R \to -\lambda_e\left(\bar{\mathcal{E}}_L e^{-i\alpha^a t_L^a} e^{-i\beta Y_L} \cdot e^{i\beta Y} e^{i\alpha^a t_L^a}\phi\right) e^{i\beta Y_R} e^{i\alpha^a t_R^a} e_R = -\lambda_e(\bar{\mathcal{E}}_L \cdot \phi)e_R,$$

since

$$-t_L^a + t_L^a + t_R^a = -\,\sigma^a/2 + \sigma^a/2 + 0 = 0, \qquad -Y_L + Y + Y_R = -(-\,1/2) + 1/2 + (-1) = 0.$$

19.6.3 Mass Term for the Up Quark

If we plug in $\phi = \phi_0 + \phi'$ into

$$-\lambda_u \epsilon^{ab}\bar{q}_L^a \phi_b^\dagger u_R + \text{h.c.},$$

where $\phi_0 = (0, v)/\sqrt{2} \Longrightarrow \phi_{0b}^\dagger = v\delta_{b2}/\sqrt{2}$ and ϵ^{ab} is totally antisymmetric, we find

$$\begin{aligned}
-\frac{\lambda_u v}{\sqrt{2}}\epsilon^{ab}\bar{q}_L^a\delta_{b2}u_R + \text{h.c.} + \mathcal{O}((\text{fields})^3) &= -\frac{\lambda_u v}{\sqrt{2}}\epsilon^{a2}\bar{q}_L^a u_R + \text{h.c.} + \mathcal{O}((\text{fields})^3) \\
&= -\frac{\lambda_u v}{\sqrt{2}}\bar{q}_L^1 u_R + \text{h.c.} + \mathcal{O}((\text{fields})^3) = -\frac{\lambda_u v}{\sqrt{2}}\bar{u}_L u_R + \text{h.c.} + \mathcal{O}((\text{fields})^3).
\end{aligned}$$

19.7 The Higgs Boson

19.7.1 The Mass of the Higgs Boson

Consider the Lagrangian

$$\mathcal{L} = \left|D_\mu \phi\right|^2 + \underbrace{\mu^2 \phi^\dagger \phi - \lambda\left(\phi^\dagger \phi\right)^2}_{=:-V}$$

and let us write the field $\phi(x)$ in the form

$$\phi(x) = U(x)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}.$$

Here, $U(x)$ is a general SU(2) unitary transformation (that is, $U^\dagger U = 1$) and $h(x) \in \mathbb{R}$ a deviation from the vacuum expectation value v, that minimizes the potential.[1] That is, we can give v as follows:

$$-\frac{\partial V}{\partial \phi^\dagger} = \mu^2 \phi - 2\lambda\left(\phi^\dagger \phi\right)\phi \stackrel{!}{=} 0 \qquad \Longrightarrow \qquad \phi_0^\dagger \phi_0 = \frac{v^2}{2} \stackrel{!}{=} \frac{\mu^2}{2\lambda} \qquad \Longleftrightarrow \qquad v \stackrel{!}{=} \frac{\mu}{\sqrt{\lambda}}.$$

Since $h(x)$ is arbitrary and $U(x) = e^{i\alpha^a(x)t^a}$ is an arbitrary unitary transformation (that is $\alpha^a(x)$ is an arbitrary function), the parameterization of the field $\phi(x)$ above in terms of $U(x)$ and $h(x)$ does not lack any generality. Since the Lagrangian is invariant under SU(2) transformations, we can perform a gauge transformation $\phi \to U^\dagger \phi$ to eliminate U from the Lagrangian.

After this gauge transformation, the potential V can be expanded like (dropping irrelevant constant terms)

$$\begin{aligned}
-V &= \mu^2 \phi^\dagger \phi - \lambda\left(\phi^\dagger \phi\right)^2 \\
&= \frac{\mu^2}{2}(v+h)^2 - \frac{\lambda}{4}(v+h)^4 \\
&= \frac{\mu^2}{2}\left(\frac{\mu}{\sqrt{\lambda}} + h\right)^2 - \frac{\lambda}{4}\left(\frac{\mu}{\sqrt{\lambda}} + h\right)^4 \\
&= -\mu^2 h^2 - \mu\sqrt{\lambda} h^3 - \frac{\lambda}{4} h^4 + \frac{\mu^4}{4\lambda} \\
&= -\frac{m_h^2}{2} h^2 - \frac{m_h}{\sqrt{2}}\sqrt{\lambda} h^3 - \frac{\lambda}{4} h^4,
\end{aligned}$$

where we introduced the mass $m_h \coloneqq \sqrt{2}\mu = \sqrt{2\lambda} v$ in the last step.

19.7.2 Coupling to Gauge Bosons

Recall from (>19.4.2), that the covariant derivative

$$D_\mu = \partial_\mu + igA_\mu^a t^a + \frac{ig'}{2} B_\mu$$

leads to

$$\left|D_\mu \phi\right|^2 = \phi^\dagger\left(\overleftarrow{\partial}_\mu \partial^\mu\right)\phi + \phi^\dagger\left(\overleftarrow{\partial}_\mu igA_b^\mu t^b + \overleftarrow{\partial}_\mu\, ig' B^\mu/2 - igA_\mu^a t^a \partial^\mu - ig' B_\mu/2\, \partial^\mu\right)\phi$$

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[1] This parametrization is completely general. Using $U(x) = \exp(i\phi'^a t^a)$ with $a = 1, 2, 3$ and the SU(2) generators $t^a = \sigma^a/2$ as well as the abbreviation $\phi_4(x) \coloneqq v + h(x)$, we find

$$\begin{aligned}
U\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \phi_4 \end{pmatrix} &= \left(1 + i\phi'^a t^a + \mathcal{O}(\phi'^2)\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \phi_4 \end{pmatrix} \\
&= \left(1 + i\phi'^1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\phi'^2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + i\phi'^3\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathcal{O}(\phi'^2)\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \phi_4 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 0 \\ \phi_4 \end{pmatrix} + i\phi'^1\begin{pmatrix} \phi_4 \\ 0 \end{pmatrix} + i\phi'^2\begin{pmatrix} -i\phi_4 \\ 0 \end{pmatrix} + i\phi'^3\begin{pmatrix} 0 \\ -\phi_4 \end{pmatrix} + \mathcal{O}(\phi'^2)\right) \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} i\phi'^1\phi_4 + \phi'^2\phi_4 \\ \phi_4 - i\phi'^3\phi_4 \end{pmatrix} + \mathcal{O}(\phi'^2) = \frac{1}{\sqrt{2}}\begin{pmatrix} -i\phi_1 - \phi_2 \\ \phi_4 + i\phi_3 \end{pmatrix} + \mathcal{O}(\phi'^2).
\end{aligned}$$

In the last step, we renamed the fields according to

$$\phi'^1\phi_4 = -\phi_1, \qquad \phi'^2\phi_4 = -\phi_2, \qquad \phi'^3\phi_4 = -\phi_3.$$

In this way, we can absorb the Goldstone boson fields ϕ^1, ϕ^2, ϕ^3 into U.

$$+\ \phi^\dagger\big(g^2 t^a t^b A^a_\mu A^\mu_b + g g' t^a A^a_\mu B^\mu + g'^2 B_\mu B^\mu/4\big)\phi.$$

For the investigation of the masses of the gauge bosons, it is sufficient to consider the last term only, due to the Higgs mechanics of section 19.3 (a few more words about that can be found in (>19.4.2)). Back in (>19.4.2), we expanded $\phi(x) = \phi_0 + \phi'(x)$ with $\phi_0 = (0, v)/\sqrt{2}$ and then kept only the terms that did not contain ϕ'. Let us now see what happens with those terms neglected in (>19.4.2) for our present "special case"[1] $\phi'(x) = \big(0, h(x)\big)/\sqrt{2}$. For this purpose, let us abbreviate the bracket of the third term as X_{ij}, where i and j are the indices of the components of the matrices $t^a \to t^a_{ij}$. Then, since X_{ij} does not contain any derivative,

$$\begin{aligned}\phi_i^\dagger X_{ij}\phi_j &= \frac{1}{2}(0, v+h)_i X_{ij}\begin{pmatrix}0\\ v+h\end{pmatrix}_j = \frac{1}{2}(v+h)\delta_{i2}X_{ij}(v+h)\delta_{j2} = \frac{1}{2}(v+h)X_{22}(v+h)\\ &= \frac{1}{2}v^2 X_{22}\left(1+\frac{h}{v}\right)^2.\end{aligned}$$

If we neglect terms containing ϕ' or h respectively, we obviously just get $\phi_i^\dagger X_{ij}\phi_j = v^2 X_{22}/2$. This is exactly what we did in (>19.4.2). Does, we can copy our result from back then and find

$$\begin{aligned}|D_\mu\phi|^2 &= \cdots + \frac{1}{8}v^2\left(g^2\big(A^1_\mu\big)^2 + g^2\big(A^2_\mu\big)^2 + \big(-gA^3_\mu + g'B_\mu\big)^2\right)\cdot\left(1+\frac{h}{v}\right)^2\\ &= \cdots + \left(m_W^2 W^-_\mu W^{+\mu} + \frac{m_Z^2}{2}\big(Z^0_\mu\big)^2\right)\cdot\left(1+\frac{h}{v}\right)^2,\end{aligned}$$

where we used the result from (>19.4.5) in the last step.

19.7.3 Coupling to Fermions

Using the form of the field $\phi(x)$ from (>19.7.1), the fermion mass terms from section 18.6 to *all* others in the fields read

$$-\lambda_e(\bar{E}_L\cdot\phi)e_R + \text{h.c.} = -\frac{\lambda_e}{\sqrt{2}}\bar{e}_L(v+h)e_R + \text{h.c.} = -m\bar{e}_L e_R\left(1+\frac{h}{v}\right) + \text{h.c.},$$

$$-\lambda_d(\bar{q}_L\cdot\phi)d_R + \text{h.c.} = -\frac{\lambda_d}{\sqrt{2}}\bar{d}_L(v+h)d_R + \text{h.c.} = -m_d\bar{d}_L d_R\left(1+\frac{h}{v}\right) + \text{h.c.},$$

$$-\lambda_u\epsilon^{ab}\bar{q}^a_L\phi^\dagger_b u_R + \text{h.c.} = -\frac{\lambda_u}{\sqrt{2}}\epsilon^{a2}\bar{q}^a_L(v+h)u_R + \text{h.c.} = -m_u\bar{u}_L u_R\left(1+\frac{h}{v}\right) + \text{h.c.}$$

19.8 Generalization to Three Generations

19.8.1 Mass Terms of Several Quark Generations

In section 18.6 we constructed the mass term for the down quark as follows:

$$-\lambda_d(\bar{q}_L\cdot\phi)d_R + \text{h.c.}$$

We now also want to include the other quark flavours, the down-types of which are s and b in addition to d. Let us write a vector $\mathcal{d} = (d, s, b)$ (and later similarly a vector $\mathcal{u} = (u, c, t)$ for the up-type quarks). Hence, q_L becomes a vector of vectors $\mathcal{q}_L$:

$$\mathcal{q}_L = \begin{pmatrix}\mathcal{u}_L\\ \mathcal{d}_L\end{pmatrix} = \begin{pmatrix}(u_L, c_L, t_L)\\ (d_L, s_L, b_L)\end{pmatrix}.$$

[1] Actually, this case is not so special as it seems, as argued in (>19.7.1), but rather quite general.

In general, λ_d becomes a matrix λ_d, providing different values for different flavour combinations in the mass term, using $\phi = (0, X)$:

$$-(\bar{q}_L \cdot \phi)\lambda_d d_R = -\lambda_d^{ij}(\bar{q}_L^i \cdot \phi)d_R^j = -X\,\bar{d}_L^i \lambda_d^{ij} d_R^j = -X\,\bar{d}_L \lambda_d d_R, \qquad i,j = 1,2,3.$$

Since $\lambda_d \lambda_d^\dagger$ and $\lambda_d^\dagger \lambda_d$ are – per definition – Hermitian matrices, we can write them in terms of a real diagonal matrix D_d^2 and certain unitary matrices S_d, R_d as[1]

$$\lambda_d \lambda_d^\dagger = S_d D_d^2 S_d^\dagger, \qquad \lambda_d^\dagger \lambda_d = R_d D_d^2 R_d^\dagger \qquad \Longrightarrow \qquad \lambda_d = S_d D_d R_d^\dagger.$$

It can easily be checked that the last expression for λ_d fulfils the two equations on the lef-thand side. Using this expression together with the basis change

$$d_R \to R_d d_R, \qquad d_L \to S_d d_L,$$

the mass term can be given as

$$-X\,\bar{d}_L \lambda_d d_R = -X\,\bar{d}_L\big(S_d D_d R_d^\dagger\big)d_R \to -X\,\bar{d}_L S_d^\dagger\big(S_d D_d R_d^\dagger\big)R_d d_R = -X\,\bar{d}_L D_d d_R = -X\,\bar{d}_L^i D_d^{ij} d_R^j.$$

Using $X = (v+h)/\sqrt{2}$ as in section 18.7 as well as $m_d^i = D_d^{ii} v/\sqrt{2}$ (no sum over i), we find the standard form of the mass terms with the Higgs boson couplings:

$$-X\,\bar{d}_L^i D_d^{ij} d_R^j = -\frac{v}{\sqrt{2}} \sum_i \bar{d}_L^i D_d^{ii} d_R^i \left(1 + \frac{h}{v}\right) = -\sum_i m_d^i \bar{d}_L^i d_R^i \left(1 + \frac{h}{v}\right).$$

Exactly the same can be done for the up-type quarks; just replace every index d with a u (for example, $R_d \to R_u$) and we find for the up-type mass term of section 18.6

$$-\epsilon^{ab}\bar{q}_L^a \phi_b^\dagger \lambda_u u_R = -\epsilon^{ab}\bar{q}_L^a X \delta_{b2} \lambda_u u_R = -X\,\bar{u}_L \lambda_u u_R = \cdots = -\sum_i m_u^i \bar{u}_L^i u_R^i \left(1 + \frac{h}{v}\right).$$

19.8.2 2D CKM Matrix

If we consider only two generations, the vectors of quark flavours have two components, $u = (u, c)$ and $d = (d, s)$. Hence, the CKM matrix V is a 2×2 matrix. It is unitary and generally complex.

A complex 2×2 matrix has eight parameters (one real and one imaginary for each of the four components). The unitarity condition

$$VV^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & ac^* + bd^* \\ ca^* + db^* & |c|^2 + |d|^2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which are actually four individual conditions, one for each component, reduces the number of parameters by four. The most general form of V can then be given as

$$V = \begin{pmatrix} \cos\theta_c\, e^{i\alpha} & \sin\theta_c\, e^{i\beta} \\ -\sin\theta_c\, e^{i(\alpha+\gamma)} & \cos\theta_c\, e^{i(\beta+\gamma)} \end{pmatrix}.$$

We can further simplify this matrix by getting rid of the phases if we rescale the fields appropriately. Consider for example the term $\bar{u}_L V d_L$. Using the rescaling

$$\bar{c}_L \to e^{-i\gamma}\bar{c}_L, \qquad d_L \to e^{-i\alpha} d_L, \qquad s_L \to e^{-i\beta} s_L,$$

the phases simply vanish:

[1] Note that the matrix AB has the same eigenvalues as the matrix BA (but different eigenvectors):

$$AB\vec{x} = \lambda\vec{x} \qquad \Leftrightarrow \qquad AB\vec{x} = \lambda\, B^{-1}B\vec{x} \qquad \Leftrightarrow \qquad BA(B\vec{x}) = \lambda(B\vec{x}).$$

$$\bar{u}_L V d_L = (\bar{u}_L, \bar{c}_L)\begin{pmatrix} \cos\theta_c\, e^{i\alpha} & \sin\theta_c\, e^{i\beta} \\ -\sin\theta_c\, e^{i(\alpha+\gamma)} & \cos\theta_c\, e^{i(\beta+\gamma)} \end{pmatrix}\begin{pmatrix} d_L \\ s_L \end{pmatrix} \to (\bar{u}_L, \bar{c}_L)\begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}\begin{pmatrix} d_L \\ s_L \end{pmatrix}.$$

19.8.3 Three Generations of Leptons

For three generations of leptons, we promote the quantities from section 18.6 as follows:

$$e \to \mathscr{e} = (e, \mu, \tau), \qquad \nu \to \mathscr{n} = (\nu_e, \nu_\mu, \nu_\tau), \qquad E_L \to \mathcal{E}_L = \begin{pmatrix} \mathscr{n}_L \\ \mathscr{e}_L \end{pmatrix} = \begin{pmatrix} (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}) \\ (e_L, \mu_L, \tau_L) \end{pmatrix}.$$

The mass term from section 18.6 then reads, using $\phi = (0, X)$,

$$-(\bar{\mathcal{E}}_L \cdot \phi)\lambda_e \mathscr{e}_R = -\lambda_e^{ij}(\bar{\mathcal{E}}_L^i \cdot \phi)\mathscr{e}_R^j = -X\,\bar{\mathscr{e}}_L^i \lambda_e^{ij} \mathscr{e}_R^j = -X\,\bar{\mathscr{e}}_L \lambda_e \mathscr{e}_R, \qquad i,j = 1,2,3.$$

As derived in (>19.8.1), we can write

$$\lambda_e = S_e D_e R_e^\dagger$$

and then change the basis of the fields according to

$$\mathscr{e}_R \to R_e \mathscr{e}_R, \qquad \mathscr{e}_L \to S_e \mathscr{e}_L.$$

This yields for the mass term

$$-X\,\bar{\mathscr{e}}_L \lambda_e \mathscr{e}_R \to -X\,\bar{\mathscr{e}}_L S_e^\dagger (S_e D_e R_e^\dagger) R_e \mathscr{e}_R = -X\,\bar{\mathscr{e}}_L D_e \mathscr{e}_R = -X \sum_i \bar{\mathscr{e}}_L^i D_e^{ii} \mathscr{e}_R^i.$$

The last step is valid, since we know from (>19.8.1), that $\mathcal{D}_e$ is diagonal. Using $X = (v+h)/\sqrt{2}$ as well as $m_e^i = D_e^{ii} v/\sqrt{2}$ we find the usual form of the fermion mass term:

$$-X \sum_i \bar{\mathscr{e}}_L^i D_e^{ii} \mathscr{e}_R^i = -\frac{v}{\sqrt{2}}\left(1 + \frac{h}{v}\right)\sum_i \bar{\mathscr{e}}_L^i D_e^{ii} \mathscr{e}_R^i = -\left(1 + \frac{h}{v}\right)\sum_i m_e^i \bar{\mathscr{e}}_L^i \mathscr{e}_R^i.$$

If we also change the neutrino fields according to

$$\mathscr{n}_L \to S_e \mathscr{n}_L,$$

we can write the change of $\mathscr{n}_L$ and $\mathscr{e}_L$ combined as

$$\begin{pmatrix} \mathscr{n}_L \\ \mathscr{e}_L \end{pmatrix} = \mathcal{E}_L \to S_e \mathcal{E}_L = \begin{pmatrix} S_e \mathscr{n}_L \\ S_e \mathscr{e}_L \end{pmatrix}.$$

Thereby, and in contrast to the case of the quarks, the matrices S_e and R_e vanish in *all* terms of the Lagrangian. We have shown it for the mass terms above, it is obvious for the kinetic terms ($\bar{\mathcal{E}}_L i\partial\!\!\!/ \mathcal{E}_L$ and $\bar{\mathscr{e}}_R i\partial\!\!\!/ \mathscr{e}_R$) and it is also obvious for the currents J_Z^μ and J_{EM}^μ from section 18.5. It is almost as obvious for he currents $J_W^{\pm\mu}$, for example

$$J_W^{+\mu} = -\frac{1}{\sqrt{2}}\bar{\mathscr{n}}_L \gamma^\mu \mathscr{e}_L \quad \to \quad -\frac{1}{\sqrt{2}}\bar{\mathscr{n}}_L S_e^\dagger \gamma^\mu S_e \mathscr{e}_L = -\frac{1}{\sqrt{2}}\bar{\mathscr{n}}_L \gamma^\mu \mathscr{e}_L.$$

Note, that the reason why this works is, that we did not introduce a separate unitary matrix S_n for the neutrinos, but used S_e again. On the other hand, for the quarks we used different matrices S_u and S_d for up- and down-type quarks, which lead us to CKM mixing. There is no better reason for those different ways of treating leptons and quarks than experimental evidence: We know from experiments, that quark generations mix and that lepton generations do not.

19.9 Overview of Electroweak Theory

19.9.1 The Electroweak Lagrangian

Let us denote the complete electroweak Lagrangian as

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{\psi} i \not{D} \psi + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yuk}}.$$

$-F^a_{\mu\nu} F^{\mu\nu}_a/4$ contains the kinetic and interaction terms among the gauge bosons ($W^\pm, Z^0$ and photon). $\bar{\psi} i \not{D} \psi$ contains the kinetic terms of the fermions and their interactions with gauge bosons. $\mathcal{L}_{\text{Higgs}}$ contains the kinetic term and the potential of the Higgs field as well as the interaction of the Higgs field with the gauge bosons (and hence also the gauge boson mass terms). And finally, the fermion mass terms are also called *Yukawa couplings*, which is why we group them together as $\mathcal{L}_{\text{Yuk}}$. Note, that this is not the complete Lagrangian of the standard model, since QCD terms are missing.

19.9.2 Gauge Boson Terms

By "gauge boson terms", we mean

$$-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a,$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu =: \tilde{F}^a_{\mu\nu} + \tilde{f}^a_{\mu\nu}, \qquad \tilde{F}^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu, \quad \tilde{f}^a_{\mu\nu} := -g f^{abc} A^b_\mu A^c_\nu.$$

Here, we want to include the field that we called B_μ in section 19.4, such that $a = 1, 2, 3, 4$ and $B_\mu = A^4_\mu$. Since $B_\mu = A^a_\mu$ is an U(1) gauge field, whose structure constants are zero, we define f^{abc} to be the totally antisymmetric tensor, except any of its indices is 4; in this case, f^{abc} shall be zero.[1]

Let us now express this gauge boson term in terms of the mass eigenstates $W^\pm_\mu, Z_\mu, A_\mu$, using the formulas

$$A^3_\mu = \sin\theta_W A_\mu + \cos\theta_W Z_\mu, \qquad A^4_\mu = B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu,$$

$$A^1_\mu = \frac{1}{\sqrt{2}}\left(W^-_\mu + W^+_\mu\right), \qquad A^2_\mu = \frac{1}{\sqrt{2}i}\left(W^-_\mu - W^+_\mu\right).$$

TERMS WITH $a = 3$ AND $a = 4$:

Let us first consider the terms with $a = 3$ and $a = 4$:

$$F^3_{\mu\nu} F^{\mu\nu}_3 + F^4_{\mu\nu} F^{\mu\nu}_4 = \tilde{F}^3_{\mu\nu} \tilde{F}^{\mu\nu}_3 + \tilde{F}^4_{\mu\nu} \tilde{F}^{\mu\nu}_4 + 2\tilde{F}^3_{\mu\nu} \tilde{f}^{\mu\nu}_3 + \tilde{f}^3_{\mu\nu} \tilde{f}^{\mu\nu}_3.$$

Note that $\tilde{f}^4_{\mu\nu} = 0$. First, we compute

$$\tilde{F}^3_{\mu\nu} = \partial_\mu(\sin\theta_W A_\nu + \cos\theta_W Z_\nu) - \partial_\nu\left(\sin\theta_W A_\mu + \cos\theta_W Z_\mu\right) = \sin\theta_W F_{\mu\nu} + \cos\theta_W Z_{\mu\nu},$$

$$\tilde{F}^4_{\mu\nu} = \partial_\mu(\cos\theta_W A_\nu - \sin\theta_W Z_\nu) - \partial_\nu\left(\cos\theta_W A_\mu - \sin\theta_W Z_\mu\right) = \cos\theta_W F_{\mu\nu} - \sin\theta_W Z_{\mu\nu}$$

where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ and $Z_{\mu\nu} := \partial_\mu Z_\nu - \partial_\nu Z_\mu$. Thus, we find that

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[1] Instead, we could also use

$$-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

instead of $-F^a_{\mu\nu} F^{\mu\nu}_a/4$ only as the gauge boson term in the Lagrangian. If we do so, then a takes on only the values $a = 1, 2, 3$ and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$\tilde{F}^3_{\mu\nu}\tilde{F}_3^{\mu\nu} + \tilde{F}^4_{\mu\nu}\tilde{F}_4^{\mu\nu} = F_{\mu\nu}F^{\mu\nu} + Z_{\mu\nu}Z^{\mu\nu}.$$

Okay, we have turned the pure-$\tilde{F}$-terms into terms of the mass eigenstates. Let us now tackle the remaining two terms, containing an $\tilde{f}$:

$$\begin{aligned}\tilde{f}^3_{\mu\nu} &= -gf^{3bc}A^b_\mu A^c_\nu = -g\big(A^1_\mu A^2_\nu - A^2_\mu A^1_\nu\big)\\ &= -\frac{g}{2i}\Big(\big(W^-_\mu + W^+_\mu\big)(W^-_\nu - W^+_\nu) - \big(W^-_\mu - W^+_\mu\big)(W^-_\nu + W^+_\nu)\Big) = ig\big(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu\big).\end{aligned}$$

Hence, we find

$$\begin{aligned}2\tilde{F}^3_{\mu\nu}\tilde{f}_3^{\mu\nu} &= 2ig\big(\sin\theta_w F_{\mu\nu} + \cos\theta_w Z_{\mu\nu}\big)(W^{+\mu}W^{-\nu} - W^{-\mu}W^{+\nu})\\ &= 4ig\big(\sin\theta_w F_{\mu\nu}W^{+\mu}W^{-\nu} + \cos\theta_w Z_{\mu\nu}W^{+\mu}W^{-\nu}\big),\end{aligned}$$

where we used $F_{\mu\nu} = -F_{\nu\mu}$ such that $F_{\mu\nu}W^{-\mu}W^{+\nu} = -F_{\mu\nu}W^{+\mu}W^{-\nu}$. Finally,

$$\begin{aligned}\tilde{f}^3_{\mu\nu}\tilde{f}_3^{\mu\nu} &= -g^2\big(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu\big)(W^{+\mu}W^{-\nu} - W^{-\mu}W^{+\nu})\\ &= -2g^2\big(W^+_\mu W^-_\nu W^{+\mu}W^{-\nu} - W^+_\mu W^-_\nu W^{-\mu}W^{+\nu}\big) = -2g^2\left(\big(W^+_\mu\big)^2(W^-_\nu)^2 - \big(W^+_\mu W^{-\mu}\big)^2\right).\end{aligned}$$

Putting everything together, we find

$$\begin{aligned}-\frac{1}{4}\big(F^3_{\mu\nu}F_3^{\mu\nu} + F^4_{\mu\nu}F_4^{\mu\nu}\big) &= -\frac{1}{4}\big(\tilde{F}^3_{\mu\nu}\tilde{F}_3^{\mu\nu} + \tilde{F}^4_{\mu\nu}\tilde{F}_4^{\mu\nu} + 2\tilde{F}^3_{\mu\nu}\tilde{f}_3^{\mu\nu} + \tilde{f}^3_{\mu\nu}\tilde{f}_3^{\mu\nu}\big)\\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}\\ &\quad - ig\big(\sin\theta_w F_{\mu\nu}W^{+\mu}W^{-\nu} + \cos\theta_w Z_{\mu\nu}W^{+\mu}W^{-\nu}\big)\\ &\quad + \frac{g^2}{2}\Big(\big(W^+_\mu\big)^2(W^-_\nu)^2 - \big(W^+_\mu W^{-\mu}\big)^2\Big).\end{aligned}$$

TERMS WITH $a = 1$ AND $a = 2$:

Let us now consider the terms with $a = 1$ and $a = 2$:

$$F^1_{\mu\nu}F_1^{\mu\nu} + F^2_{\mu\nu}F_2^{\mu\nu} = \tilde{F}^1_{\mu\nu}\tilde{F}_1^{\mu\nu} + \tilde{F}^2_{\mu\nu}\tilde{F}_2^{\mu\nu} + 2\tilde{F}^1_{\mu\nu}\tilde{f}_1^{\mu\nu} + 2\tilde{F}^2_{\mu\nu}\tilde{f}_2^{\mu\nu} + \tilde{f}^1_{\mu\nu}\tilde{f}_1^{\mu\nu} + \tilde{f}^2_{\mu\nu}\tilde{f}_2^{\mu\nu}.$$

Consider

$$\tilde{F}^1_{\mu\nu} = \frac{1}{\sqrt{2}}\Big(\partial_\mu(W^-_\nu + W^+_\nu) - \partial_\nu\big(W^-_\mu + W^+_\mu\big)\Big) = \frac{1}{\sqrt{2}}\big(W^-_{\mu\nu} + W^+_{\mu\nu}\big),$$

$$\tilde{F}^2_{\mu\nu} = \frac{1}{\sqrt{2}i}\Big(\partial_\mu(W^-_\nu - W^+_\nu) - \partial_\nu\big(W^-_\mu - W^+_\mu\big)\Big) = \frac{1}{\sqrt{2}i}\big(W^-_{\mu\nu} - W^+_{\mu\nu}\big),$$

where $W^\pm_{\mu\nu} := \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu$. Thus, we find

$$\begin{aligned}\tilde{F}^1_{\mu\nu}\tilde{F}_1^{\mu\nu} + \tilde{F}^2_{\mu\nu}\tilde{F}_2^{\mu\nu} &= \frac{1}{2}\Big(\big(W^-_{\mu\nu} + W^+_{\mu\nu}\big)(W^{-\mu\nu} + W^{+\mu\nu}) - \big(W^-_{\mu\nu} - W^+_{\mu\nu}\big)(W^{-\mu\nu} - W^{+\mu\nu})\Big)\\ &= 2W^-_{\mu\nu}W^{+\mu\nu}.\end{aligned}$$

Okay, we have turned the pure-$\tilde{F}$-terms into terms of the mass eigenstates. Let us now tackle the remaining four terms, containing an $\tilde{f}$:

$$\begin{aligned}\tilde{f}^1_{\mu\nu} &= -gf^{1bc}A^b_\mu A^c_\nu = -g\big(A^2_\mu A^3_\nu - A^3_\mu A^2_\nu\big)\\ &= -\frac{g}{\sqrt{2}i}\Big(\big(W^-_\mu - W^+_\mu\big)(\sin\theta_w A_\nu + \cos\theta_w Z_\nu) - \big(\sin\theta_w A_\mu + \cos\theta_w Z_\mu\big)(W^-_\nu - W^+_\nu)\Big)\\ &= -\frac{g\cos\theta_w}{\sqrt{2}i}\big(\widetilde{W}^-_\mu Z_\nu - Z_\mu\widetilde{W}^-_\nu\big) - \frac{g\sin\theta_w}{\sqrt{2}i}\big(\widetilde{W}^-_\mu A_\nu - A_\mu\widetilde{W}^-_\nu\big),\end{aligned}$$

$$
\begin{aligned}
\tilde{f}^2_{\mu\nu} &= -gf^{2bc}A^b_\mu A^c_\nu = -g\left(A^3_\mu A^1_\nu - A^1_\mu A^3_\nu\right)\\
&= -\frac{g}{\sqrt{2}}\Big(\left(\sin\theta_w A_\mu + \cos\theta_w Z_\mu\right)\left(W^-_\nu + W^+_\nu\right) - \left(W^-_\mu + W^+_\mu\right)\left(\sin\theta_w A_\nu + \cos\theta_w Z_\nu\right)\Big)\\
&= -\frac{g\sin\theta_w}{\sqrt{2}}\left(A_\mu\widetilde{W}^+_\nu - \widetilde{W}^+_\mu A_\nu\right) - \frac{g\cos\theta_w}{\sqrt{2}}\left(Z_\mu\widetilde{W}^+_\nu - \widetilde{W}^+_\mu Z_\nu\right),
\end{aligned}
$$

where $\widetilde{W}^\pm_\mu \coloneqq W^-_\mu \pm W^+_\mu$. Thus, we find

$$
\begin{aligned}
2\tilde{F}^1_{\mu\nu}\tilde{f}^{\mu\nu}_1 &= ig\left(W^-_{\mu\nu} + W^+_{\mu\nu}\right)\Big(\cos\theta_w\left(\widetilde{W}^{-\mu}Z^\nu - Z^\mu\widetilde{W}^{-\nu}\right) + \sin\theta_w\left(\widetilde{W}^{-\mu}A^\nu - A^\mu\widetilde{W}^{-\nu}\right)\Big)\\
&= ig\cos\theta_w\left(W^-_{\mu\nu} + W^+_{\mu\nu}\right)\left(\widetilde{W}^{-\mu}Z^\nu - Z^\mu\widetilde{W}^{-\nu}\right) + ig\sin\theta_w\left(W^-_{\mu\nu} + W^+_{\mu\nu}\right)\left(\widetilde{W}^{-\mu}A^\nu - A^\mu\widetilde{W}^{-\nu}\right)\\
&= 2ig\cos\theta_w\left(W^-_{\mu\nu} + W^+_{\mu\nu}\right)\widetilde{W}^{-\mu}Z^\nu + 2ig\sin\theta_w\left(W^-_{\mu\nu} + W^+_{\mu\nu}\right)\widetilde{W}^{-\mu}A^\nu,
\end{aligned}
$$

$$
\begin{aligned}
2\tilde{F}^2_{\mu\nu}\tilde{f}^{\mu\nu}_2 &= ig\left(W^-_{\mu\nu} - W^+_{\mu\nu}\right)\Big(\sin\theta_w\left(A^\mu\widetilde{W}^{+\nu} - \widetilde{W}^{+\mu}A^\nu\right) + \cos\theta_w\left(Z^\mu\widetilde{W}^{+\nu} - \widetilde{W}^{+\mu}Z^\nu\right)\Big)\\
&= ig\sin\theta_w\left(W^-_{\mu\nu} - W^+_{\mu\nu}\right)\left(A^\mu\widetilde{W}^{+\nu} - \widetilde{W}^{+\mu}A^\nu\right) + ig\cos\theta_w\left(W^-_{\mu\nu} - W^+_{\mu\nu}\right)\left(Z^\mu\widetilde{W}^{+\nu} - \widetilde{W}^{+\mu}Z^\nu\right)\\
&= 2ig\sin\theta_w\left(W^-_{\mu\nu} - W^+_{\mu\nu}\right)A^\mu\widetilde{W}^{+\nu} + 2ig\cos\theta_w\left(W^-_{\mu\nu} - W^+_{\mu\nu}\right)Z^\mu\widetilde{W}^{+\nu}.
\end{aligned}
$$

Here, we used $W^\pm_{\mu\nu} = -W^\pm_{\nu\mu}$. and hence $W^\pm_{\mu\nu}X^{\mu\nu} = -W^\pm_{\mu\nu}X^{\nu\mu}$ for any tensor $X^{\mu\nu}$. Using this identity again, we can further simplify the sum of the two expressions above:

$$
\begin{aligned}
&2\tilde{F}^1_{\mu\nu}\tilde{f}^{\mu\nu}_1 + 2\tilde{F}^2_{\mu\nu}\tilde{f}^{\mu\nu}_2\\
&\qquad= 2ig\cos\theta_w\Big(\left(W^-_{\mu\nu} + W^+_{\mu\nu}\right)\widetilde{W}^{-\mu}Z^\nu + \left(W^-_{\mu\nu} - W^+_{\mu\nu}\right)Z^\mu\widetilde{W}^{+\nu}\Big)\\
&\qquad\quad + 2ig\sin\theta_w\Big(\left(W^-_{\mu\nu} + W^+_{\mu\nu}\right)\widetilde{W}^{-\mu}A^\nu + \left(W^-_{\mu\nu} - W^+_{\mu\nu}\right)A^\mu\widetilde{W}^{+\nu}\Big)\\
&\qquad= 2ig\left(\cos\theta_w Z^\nu + \sin\theta_w A^\nu\right)\Big(\left(W^-_{\mu\nu} + W^+_{\mu\nu}\right)\widetilde{W}^{-\mu} - \left(W^-_{\mu\nu} - W^+_{\mu\nu}\right)\widetilde{W}^{+\mu}\Big)\\
&\qquad= -4ig\left(\cos\theta_w Z^\nu + \sin\theta_w A^\nu\right)\left(W^-_{\mu\nu}W^{+\mu} - W^+_{\mu\nu}W^{-\mu}\right)
\end{aligned}
$$

Further, we find

$$
\begin{aligned}
&\tilde{f}^1_{\mu\nu}\tilde{f}^{\mu\nu}_1\\
&\qquad= \frac{g^2}{2i^2}\cos^2\theta_w\left(\widetilde{W}^-_\mu Z_\nu - Z_\mu\widetilde{W}^-_\nu\right)^2\\
&\qquad\quad + \frac{g^2}{i^2}\cos\theta_w\sin\theta_w\left(\widetilde{W}^-_\mu Z_\nu - Z_\mu\widetilde{W}^-_\nu\right)\left(\widetilde{W}^{-\mu}A^\nu - A^\mu\widetilde{W}^{-\nu}\right)\\
&\qquad\quad + \frac{g^2}{2i^2}\sin^2\theta_w\left(\widetilde{W}^-_\mu A_\nu - A_\mu\widetilde{W}^-_\nu\right)^2,\\
&\tilde{f}^2_{\mu\nu}\tilde{f}^{\mu\nu}_2\\
&\qquad= \frac{g^2}{2}\sin^2\theta_w\left(A_\mu\widetilde{W}^+_\nu - \widetilde{W}^+_\mu A_\nu\right)^2\\
&\qquad\quad + g^2\cos\theta_w\sin\theta_w\left(A_\mu\widetilde{W}^+_\nu - \widetilde{W}^+_\mu A_\nu\right)\left(Z^\mu\widetilde{W}^{+\nu} - \widetilde{W}^{+\mu}Z^\nu\right)\\
&\qquad\quad + \frac{g^2}{2}\cos^2\theta_w\left(Z_\mu\widetilde{W}^+_\nu - \widetilde{W}^+_\mu Z_\nu\right)^2.
\end{aligned}
$$

The sum of the two terms above can further be simplified[1]

---

[1] For the terms multiplied by $\cos^2\theta_w$, we use

$$
\begin{aligned}
-\left(\widetilde{W}^-_\mu Z_\nu - Z_\mu\widetilde{W}^-_\nu\right)^2 + \left(Z_\mu\widetilde{W}^+_\nu - \widetilde{W}^+_\mu Z_\nu\right)^2 &= -2\left(\widetilde{W}^-_\mu Z_\nu\right)^2 + 2\left(Z_\mu\widetilde{W}^+_\nu\right)^2 + 2Z_\nu Z^\mu\widetilde{W}^-_\mu\widetilde{W}^{-\nu} - 2Z_\mu Z^\nu\widetilde{W}^+_\nu\widetilde{W}^{+\mu}\\
&= 8(Z_\nu)^2W^-_\mu W^{+\mu} + 2Z_\mu Z^\nu\widetilde{W}^-_\nu\widetilde{W}^{-\mu} - 2Z_\mu Z^\nu\widetilde{W}^+_\nu\widetilde{W}^{+\mu} = 8(Z_\nu)^2W^-_\mu W^{+\mu} - 8Z_\mu Z^\nu W^-_\nu W^{+\mu}.
\end{aligned}
$$

To get this relation, note that

$$
\begin{aligned}
\left(\widetilde{W}^\pm_\mu\right)^2 &= \left(W^-_\mu \pm W^+_\mu\right)^2 = \left(W^-_\mu\right)^2 \pm 2W^-_\mu W^{+\mu} + \left(W^+_\mu\right)^2,\\
Z_\mu Z^\nu\widetilde{W}^\pm_\nu\widetilde{W}^{\pm\mu} &= Z_\mu Z^\nu\left(W^-_\nu \pm W^+_\nu\right)\left(W^{-\mu} \pm W^{+\mu}\right) = Z_\mu Z^\nu\left(W^-_\nu W^{-\mu} \pm 2W^-_\nu W^{+\mu} + W^+_\nu W^{+\mu}\right).
\end{aligned}
$$

$$\begin{aligned}
&\tilde f^1_{\mu\nu}\tilde f_1^{\mu\nu}+\tilde f^2_{\mu\nu}\tilde f_2^{\mu\nu}\\
&\quad=\frac{g^2}{2}\cos^2\theta_w\left(8(Z_\nu)^2W^-_\mu W^{+\mu}-8Z_\mu Z^\nu W^-_\nu W^{+\mu}\right)\\
&\qquad+\frac{g^2}{2}\sin^2\theta_w\left(8(A_\nu)^2W^-_\mu W^{+\mu}-8A_\mu A^\nu W^-_\nu W^{+\mu}\right)\\
&\qquad+g^2\cos\theta_w\sin\theta_w\left(8Z_\nu A^\nu W^-_\mu W^{+\mu}-4A_\mu Z^\nu(W^-_\nu W^{+\mu}+W^+_\nu W^{-\mu})\right)\\
&\quad=4g^2\cos^2\theta_w\left((Z_\nu)^2W^-_\mu W^{+\mu}-Z_\mu Z^\nu W^-_\nu W^{+\mu}\right)\\
&\qquad+4g^2\sin^2\theta_w\left((A_\nu)^2W^-_\mu W^{+\mu}-A_\mu A^\nu W^-_\nu W^{+\mu}\right)\\
&\qquad+4g^2\cos\theta_w\sin\theta_w\left(2Z_\nu A^\nu W^-_\mu W^{+\mu}-A_\mu Z^\nu(W^-_\nu W^{+\mu}+W^+_\nu W^{-\mu})\right).
\end{aligned}$$

Putting everything together, we find

$$\begin{aligned}
&-\frac14\left(F^1_{\mu\nu}F_1^{\mu\nu}+F^2_{\mu\nu}F_2^{\mu\nu}\right)=-\frac12 W^-_{\mu\nu}W^{+\mu\nu}-\frac14\left(2\tilde F^1_{\mu\nu}\tilde f_1^{\mu\nu}+2\tilde F^2_{\mu\nu}\tilde f_2^{\mu\nu}\right)-\frac14\tilde f^1_{\mu\nu}\tilde f_1^{\mu\nu}-\frac14\tilde f^2_{\mu\nu}\tilde f_2^{\mu\nu}\\
&\quad=-\frac12 W^-_{\mu\nu}W^{+\mu\nu}\\
&\qquad+\ ig(\cos\theta_w Z^\nu+\sin\theta_w A^\nu)\left(W^-_{\mu\nu}W^{+\mu}-W^+_{\mu\nu}W^{-\mu}\right)\\
&\qquad-g^2\cos^2\theta_w\left((Z_\nu)^2W^-_\mu W^{+\mu}-Z_\mu Z^\nu W^-_\nu W^{+\mu}\right)\\
&\qquad-g^2\sin^2\theta_w\left((A_\nu)^2W^-_\mu W^{+\mu}-A_\mu A^\nu W^-_\nu W^{+\mu}\right)\\
&\qquad-g^2\cos\theta_w\sin\theta_w\left(2Z_\nu A^\nu W^-_\mu W^{+\mu}-A_\mu Z^\nu(W^-_\nu W^{+\mu}+W^+_\nu W^{-\mu})\right).
\end{aligned}$$

THE TOTAL RESULT:

When we combine the results of the $a=1,2$ and $a=3,4$ terms $-F^a_{\mu\nu}F_a^{\mu\nu}/4$, let us order them mainly according to their prefactor ($\sin\theta_w$, $\cos\theta_w$, $\cos^2\theta_w$, $\sin^2\theta_w$, $\cos\theta_w\sin\theta_w$). In the next step, let us use $g\sin\theta_w=-e$ and $g\cos\theta=-e\cot\theta$. Then we find

$$\begin{aligned}
&-\frac14 F^a_{\mu\nu}F_a^{\mu\nu}=-\frac14\left(F^1_{\mu\nu}F_1^{\mu\nu}+F^2_{\mu\nu}F_2^{\mu\nu}+F^3_{\mu\nu}F_3^{\mu\nu}+F^4_{\mu\nu}F_4^{\mu\nu}\right)\\
&\quad=-\frac14F_{\mu\nu}F^{\mu\nu}-\frac14Z_{\mu\nu}Z^{\mu\nu}-\frac12W^-_{\mu\nu}W^{+\mu\nu}\\
&\qquad-ig\sin\theta_w\left(F_{\mu\nu}W^{+\mu}W^{-\nu}-A^\nu\left(W^-_{\mu\nu}W^{+\mu}-W^+_{\mu\nu}W^{-\mu}\right)\right)\\
&\qquad-ig\cos\theta_w\left(Z_{\mu\nu}W^{+\mu}W^{-\nu}-Z^\nu\left(W^-_{\mu\nu}W^{+\mu}-W^+_{\mu\nu}W^{-\mu}\right)\right)\\
&\qquad+\frac{g^2}{2}\left(\left(W^+_\mu\right)^2(W^-_\nu)^2-\left(W^+_\mu W^{-\mu}\right)^2\right)\\
&\qquad-g^2\cos^2\theta_w\left((Z_\nu)^2W^-_\mu W^{+\mu}-Z_\mu Z^\nu W^-_\nu W^{+\mu}\right)\\
&\qquad-g^2\sin^2\theta_w\left((A_\nu)^2W^-_\mu W^{+\mu}-A_\mu A^\nu W^-_\nu W^{+\mu}\right)\\
&\qquad-g^2\cos\theta_w\sin\theta_w\left(2Z_\nu A^\nu W^-_\mu W^{+\mu}-A_\mu Z^\nu(W^-_\nu W^{+\mu}+W^+_\nu W^{-\mu})\right)\\
&\quad=-\frac14F_{\mu\nu}F^{\mu\nu}-\frac14Z_{\mu\nu}Z^{\mu\nu}-\frac12W^-_{\mu\nu}W^{+\mu\nu}\qquad\qquad\text{kinetic terms}\\
&\qquad+ie\left(F_{\mu\nu}W^{+\mu}W^{-\nu}-A^\nu\left(W^-_{\mu\nu}W^{+\mu}-W^+_{\mu\nu}W^{-\mu}\right)\right)\qquad\qquad AW^+W^-\text{ vertex}
\end{aligned}$$

For the terms multiplied by $\sin^2\theta_w$, the same relation for A_μ instead of Z_μ is useful. For the terms multiplied by $\cos\theta_w\sin\theta_w$, we need the relation

$$\begin{aligned}
&-\left(\widetilde W^-_\mu Z_\nu-Z_\mu\widetilde W^-_\nu\right)\left(\widetilde W^{-\mu}A^\nu-A^\mu\widetilde W^{-\nu}\right)+\left(A_\mu\widetilde W^+_\nu-\widetilde W^+_\mu A_\nu\right)\left(Z^\mu\widetilde W^{+\nu}-\widetilde W^{+\mu}Z^\nu\right)\\
&\quad=-2Z_\nu A^\nu\widetilde W^-_\mu\widetilde W^{-\mu}+2Z_\nu A^\mu\widetilde W^-_\mu\widetilde W^{-\nu}+2A_\mu Z^\mu\widetilde W^+_\nu\widetilde W^{+\nu}-2A_\mu Z^\nu\widetilde W^+_\nu\widetilde W^{+\mu}\\
&\quad=2Z_\nu A^\nu\left(\left(\widetilde W^+_\mu\right)^2-\left(\widetilde W^-_\mu\right)^2\right)+2A_\mu Z^\nu\left(\widetilde W^-_\nu\widetilde W^{-\mu}-\widetilde W^+_\nu\widetilde W^{+\mu}\right)\\
&\quad=8Z_\nu A^\nu W^-_\mu W^{+\mu}-4A_\mu Z^\nu(W^-_\nu W^{+\mu}+W^+_\nu W^{-\mu}).
\end{aligned}$$

$$
\begin{aligned}
&+ ie\cot\theta_w \left(Z_{\mu\nu}W^{+\mu}W^{-\nu} - Z^\nu\left(W^-_{\mu\nu}W^{+\mu} - W^+_{\mu\nu}W^{-\mu}\right)\right) && ZW^+W^- \text{ vertex}\\
&+ \frac{e^2}{2\sin^2\theta_w}\left(W^+_\mu W^{+\mu}W^-_\nu W^{-\nu} - W^+_\mu W^{-\mu}W^+_\nu W^{-\nu}\right) && W^+W^+W^-W^- \text{ vertex}\\
&+ e^2\left(A_\mu A^\nu W^-_\nu W^{+\mu} - A_\nu A^\nu W^-_\mu W^{+\mu}\right) && AAW^+W^- \text{ vertex}\\
&+ e^2\cot^2\theta_w \left(Z_\mu Z^\nu W^-_\nu W^{+\mu} - Z_\nu Z^\nu W^-_\mu W^{+\mu}\right) && ZZW^+W^- \text{ vertex}\\
&+ e^2\cot\theta_w \left(A_\mu Z^\nu(W^-_\nu W^{+\mu} + W^+_\nu W^{-\mu}) - 2Z_\nu A^\nu W^-_\mu W^{+\mu}\right). && AZW^+W^- \text{ vertex}
\end{aligned}
$$

19.9.3 Fermion Kinetic and Interaction Terms

The next part of the Lagrangian from (>19.9.1) is

$$\bar\psi i\not{D}\psi, \qquad \text{where} \qquad D_\mu = \partial_\mu + igt^a A^a_\mu.$$

Let us, again, let the index a run over $a = 1, 2, 3, 4$, such that $A^4_\mu = B_\mu$ and $t^4 = Y$. As we saw in section 19.4, this covariant derivative can be given in terms of the mass eigenstates $W^\pm_\mu, Z_\mu, A_\mu$ as

$$D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}}\left(W^+_\mu t^+ + W^-_\mu t^-\right) - \frac{ig}{\cos\theta_w}(t^3 - \sin^2\theta\, Q)Z_\mu - ieQA_\mu,$$

where $t^\pm \coloneqq t^1 \pm it^2$ and $Q = t^3 + Y$.

We must distinguish between left- and right-handed particles and we need to describe leptons as well as quarks and we know from section 19.5 that[1]

$$
\begin{aligned}
\bar\psi i\not{D}\psi &= \bar\psi_L i\not{D}\psi_L + \bar\psi_R i\not{D}\psi_R = (\bar{\mathcal{E}}_L i\not{D}e_L + \bar{q}'_L i\not{D}q'_L) + \left(\bar e_R i\not{D}e_R + \bar u'_R i\not{D}u'_R + \bar d'_R i\not{D}d'_R\right)\\
&= \bar{\mathcal{E}}_L i\not\partial\mathcal{E}_L + \bar e_R i\not\partial e_R + \bar q i\not\partial q + g\left(W^+_\mu J^{+\mu}_W + W^-_\mu J^{-\mu}_W + Z_\mu J^\mu_Z\right) + eA_\mu J^\mu_{EM}.
\end{aligned}
$$

The currents are given by

$$J^{+\mu}_W = -\frac{1}{\sqrt{2}}(\bar\nu_L\gamma^\mu e_L + \bar u_L V\gamma^\mu d_L),$$

$$J^{-\mu}_W = -\frac{1}{\sqrt{2}}\left(\bar e_L\gamma^\mu\nu_L + \bar d_L V^\dagger u_L\right),$$

$$
\begin{aligned}
J^\mu_Z = \frac{1}{\cos\theta_w}\Bigg(&\frac{1}{2}\bar\nu_L\gamma^\mu\nu_L + \left(\sin^2\theta_w - \frac{1}{2}\right)\bar e_L\gamma^\mu e_L + \sin^2\theta_w\,\bar e_R\gamma^\mu e_R + \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right)\bar u_L\gamma^\mu u_L\\
&+ \left(\frac{1}{3}\sin^2\theta_w - \frac{1}{2}\right)\bar d_L\gamma^\mu d_L - \frac{2}{3}\sin^2\theta_w\,\bar u_R\gamma^\mu u_R + \frac{1}{3}\sin^2\theta_w\,\bar d_R\gamma^\mu d_R\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
J^\mu_{EM} &= -\bar e_L\gamma^\mu e_L - \bar e_R\gamma^\mu e_R + \frac{2}{3}\bar u_L\gamma^\mu u_L - \frac{1}{3}\bar d_L\gamma^\mu d_L + \frac{2}{3}\bar u_R\gamma^\mu u_R - \frac{1}{3}\bar d_R\gamma^\mu d_R\\
&= -\bar e\gamma^\mu e + \frac{2}{3}\bar u\gamma^\mu u - \frac{1}{3}\bar d\gamma^\mu d.
\end{aligned}
$$

Here, we used the notation

$$\mathcal{E}_L \coloneqq \begin{pmatrix}\nu_L\\ e_L\end{pmatrix} \coloneqq \begin{pmatrix}(\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})\\ (e_L, \mu_L, \tau_L)\end{pmatrix}, \qquad e_R \coloneqq (e_R, \mu_R, \tau_R), \qquad e = e_L + e_R,$$

---

[1] The dashed q's are not the mass eigenstates (see section 19.8):

$$q' \coloneqq q'_L + q'_R = \begin{pmatrix}u'_L\\ d'_L\end{pmatrix} + \begin{pmatrix}u'_R\\ d'_R\end{pmatrix} = \begin{pmatrix}S_u u_L\\ S_d d_L\end{pmatrix} + \begin{pmatrix}R_u u_R\\ R_d d_R\end{pmatrix}, \qquad \text{where} \qquad S_u^\dagger S_d = V.$$

V is the CKM matrix. Note, that S_u, S_d, R_u, R_d are unitary matrixes, such that

$$
\begin{aligned}
\bar q' q' &= \bar q'_L q'_L + \bar q'_R q'_R = \left(\bar u_L S_u^\dagger, \bar d_L S_d^\dagger\right)\begin{pmatrix}S_u u_L\\ S_d d_L\end{pmatrix} + \left(\bar u_R R_u^\dagger, \bar d_R R_d^\dagger\right)\begin{pmatrix}R_u u_R\\ R_d d_R\end{pmatrix} = (\bar u_L, \bar d_L)\begin{pmatrix}u_L\\ d_L\end{pmatrix} + (\bar u_R, \bar d_R)\begin{pmatrix}u_R\\ d_R\end{pmatrix}\\
&= \bar q_L q_L + \bar q_R q_R = \bar q q.
\end{aligned}
$$

$$q_X \coloneqq \begin{pmatrix} u_X \\ d_X \end{pmatrix} \coloneqq \begin{pmatrix} (u_X, c_X, t_X) \\ (d_X, s_X, b_X) \end{pmatrix}, \qquad X = L, R, \qquad q \coloneqq q_L + q_R,$$

where left- and right-handed particles separate per definition (such that $\bar{q}q = \bar{q}_L q_L + \bar{q}_R q_R$).

19.9.4 The Higgs Sector

The Higgs sector of the Lagrangian can be given as

$$\mathcal{L} = |D_\mu \phi|^2 - V(\phi), \qquad V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2.$$

The first term contains the kinetic terms of the Higgs field as well as the interaction with the gauge boson (that includes the gauge boson mass terms). The potential contains the mass term of the Higgs boson and its self-interactions.

In section 19.3, 19.4 and 19.7, we found that

$$|D_\mu \phi|^2 = |\partial_\mu \phi|^2 + \text{Higgs mechanism terms} + \left(m_W^2 W_\mu^- W^{+\mu} + \frac{m_Z^2}{2} Z_\mu Z^\mu \right) \cdot \left(1 + \frac{h}{v}\right)^2,$$

where the Higgs mechanics terms effectively ensure the transverse structure of the propagator of the gauge bosons only (see section 19.3).

In section 19.7, we also found

$$V(\phi) = \frac{m_h^2}{2} h^2 + \frac{m_h \sqrt{\lambda}}{\sqrt{2}} h^3 + \frac{\lambda}{4} h^4.$$

19.9.5 The Yukawa Sector (Fermion Mass Terms)

The fermion mass terms that we investigated in section 19.6 and 19.8 are also called *Yukawa couplings*. In section 19.8 we found the quark mass terms

$$-(\bar{q}_L' \cdot \phi)\lambda_d d_R' - \epsilon^{ab} \bar{q}_L'^a \phi^{\dagger b} \lambda_d u_R' + \text{h.c.} = -\left(1 + \frac{h}{v}\right) \sum_i \left(m_d^i \bar{d}_L^i d_R^i + m_d^i \bar{u}_L^i u_R^i \right) + \text{h.c.},$$

where the undashed quark spinors are the mass eigenstates (see footnote on page 228).

For the lepton mass terms, we found

$$-(\bar{\mathcal{E}}_L \cdot \phi)\lambda_e e_R + \text{h.c.} = -\left(1 + \frac{h}{v}\right) \sum_i m_e^i \bar{e}_L^i e_R^i + \text{h.c.}$$

(since the unitary matrices that produce the mass eigenstates for leptons disappear from the theory, we can work in the mass eigenstates right from the beginning and do not need to use the dashed spinors as for the quarks).

19.9.6 Gauge Transformation Charges

For the sake of gauge transformations, we group the left-handed leptons and the left-handed quarks into doublets

$$\mathcal{E}_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \qquad q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix},$$

whereas we treat the right-handed leptons and quarks as singlets

$$e_R, u_R, d_R$$

(note, that there are no right-handed neutrinos ν_R). Additionally, we have to deal with the field ϕ. A local $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge transformation then corresponds to

$$\psi \to U\psi = e^{i\alpha^a(x)t^a + i\beta(x)Y}\psi$$

for arbitrary functions $\alpha^a(x), \beta(x)$. Here, ψ can be any of the five objects $\mathcal{E}_L, q_L, e_R, u_R, d_R$. Depending on these five objects, the generators t^a may have a different representation and Y may be equal to a different number. For the electroweak theory to be gauge invariant, what we need is the following charges:

		$\mathcal{E}_L$	e_R	q_L	d_R	u_R	ϕ
SU(2):	$t^a =$	$\sigma^a/2$	0	$\sigma^a/2$	0	0	$\sigma^a/2$
U(1):	$Y =$	$-1/2$	-1	$1/6$	$-1/3$	$2/3$	$1/2$

σ^a are the Pauli matrices, i. e. the fundamental representation of SU(2).[1] Thus, let us define

$$U_{\mathcal{E}L} = e^{i\alpha^a(x)t^a - i\beta(x)\cdot 1/2}, \qquad U_{eR} = e^{-i\beta(x)}, \qquad U_{qL} = e^{i\alpha^a(x)t^a + i\beta(x)\cdot 1/6},$$

$$U_{dR} = e^{-i\beta(x)\cdot 1/3}, \qquad U_{uR} = e^{i\beta(x)\cdot 2/3}, \qquad U_\phi = e^{i\alpha^a(x)t^a + i\beta(x)\cdot 1/2}.$$

In the expressions above, we have $t^a = \sigma^a/2$ always. Let us check how these charges satisfy gauge invariance.

Recalling section 3.7, before switching to mass eigenstates (and obviously before symmetry braking), the first three terms

$$-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + \bar{\psi}i\not{D}\psi + \mathcal{L}_{\text{Higgs}}$$

are trivially gauge invariant. Not so the Yukawa sector, however.

GAUGE INVARIANCE OF THE YUKAWA SECTOR:

The Yukawa sector is given by the three terms (and their Hermitian conjugates)

$$\mathcal{L}_{\text{Yuk}} = -(\bar{\mathcal{E}}_L \cdot \phi)\lambda_e e_R - (\bar{q}'_L \cdot \phi)\lambda_d d'_R - \epsilon^{ab}\bar{q}'^a_L\phi^{\dagger b}\lambda_d u'_R + \text{h.c.}$$

It straight forward to show that the first two terms are gauge invariant:

$$\begin{aligned}
-(\bar{\mathcal{E}}_L \cdot \phi)\lambda_e e_R \quad &\to \quad -\big(\bar{\mathcal{E}}_L U^\dagger_{\mathcal{E}L} \cdot U_\phi \phi\big)\lambda_e U_{eR} e_R = -\big(\bar{\mathcal{E}}_L e^{-i\alpha^a t^a + i\beta\cdot 1/2} \cdot e^{i\alpha^a t^a + i\beta\cdot 1/2}\phi\big)\lambda_e e^{-i\beta} e_R \\
&\qquad = -(\bar{\mathcal{E}}_L \cdot \phi)\lambda_e e_R, \\
-(\bar{q}'_L \cdot \phi)\lambda_d d'_R \quad &\to \quad -\big(\bar{q}'_L U^\dagger_{qL} \cdot U_\phi \phi\big)\lambda_d U_{dR} d'_R \\
&\qquad = -\big(\bar{q}'_L e^{-i\alpha^a t^a - i\beta\cdot 1/6} \cdot e^{i\alpha^a t^a + i\beta\cdot 1/2}\phi\big)\lambda_d e^{-i\beta\cdot 1/3} d'_R = -(\bar{q}'_L \cdot \phi)\lambda_d d'_R.
\end{aligned}$$

For the third term, let us investigate U(1) and SU(2) invariance separately. Setting $\alpha^a = 0$, we easily find that it is U(1) invariant:

$$\epsilon^{ab}\bar{q}'^a_L\phi^{\dagger b}\lambda_d u'_R \quad \to \quad \epsilon^{ab}\bar{q}'^a_L e^{-i\beta\cdot 1/6}e^{-i\beta\cdot 1/2}\phi^{\dagger b}\lambda_d e^{i\beta\cdot 2/3}u'_R = \epsilon^{ab}\bar{q}'^a_L\phi^{\dagger b}\lambda_d u'_R.$$

Showing SU(2) invariance is somewhat more complicated. Let us start with

$$\epsilon^{ab}\bar{q}'^a_L\phi^{\dagger b}\lambda_d u'_R \quad \to \quad \epsilon^{ab}\big(\bar{q}'_L U^\dagger_{qL}\big)^a\big(\phi^\dagger U^\dagger_\phi\big)^b\lambda_d U_{uR}u'_R = \epsilon^{ab}\bar{q}'^c_L U^{\dagger ca}_{qL}\phi^{\dagger d}U^{\dagger db}_\phi\lambda_d U_{uR}u'_R,$$

where (setting now $\beta = 0$)

[1] Note, that the "charges" of SU(N) transformations (that is, their representations) are alternatively often simply given as the dimension of its representation. That is, SU(2) charge 2 or a SU(3) charge 3 means fundamental representation, that is the 2 is equivalent to $\sigma^a/2$ and the 3 to $\lambda^a/2$. An SU(3) charge 8 means the adjoint representation. And, most confusingly, a SU(N) charge 1 is denoted for the trivial representation, where all generators are zero. However, in contrast to SU(N), the U(1) charges are *always* given as the value of Y; that is the trivial representation of U(1) is never denoted as 1 but always as 0.

$$U_\phi^{\dagger db} = \left(e^{-i\alpha^e t^e}\right)^{db} = \delta_{db} - i\alpha^e t^e_{db} + \mathcal{O}(\alpha^2),$$

$$U_{q_L}^{\dagger ca} = \left(e^{-i\alpha^f t^f}\right)^{ca} = \delta_{ca} - i\alpha^f t^f_{ca} + \mathcal{O}(\alpha^2).$$

Note, that $e, f = 1, 2, 3$, but all other indices only take on $1, 2$. Dropping terms of order α^2 (α is the function inside the exponent of U, not the fine structure constant), note that

$$\begin{aligned}\epsilon^{ab} U_\phi^{\dagger db} U_{q_L}^{\dagger ca} &= \epsilon^{ab}(\delta_{db} - i\alpha^e t^e_{db})(\delta_{ca} - i\alpha^f t^f_{ca}) = \epsilon^{ab}\left(\delta_{db}\delta_{ca} - i\alpha^f \delta_{db} t^f_{ca} - i\alpha^e \delta_{ca} t^e_{db}\right)\\ &= (\delta_{d2}\delta_{c1} - \delta_{d1}\delta_{c2}) - i\alpha^f\left(\delta_{d2} t^f_{c1} - \delta_{d1} t^f_{c2}\right) - i\alpha^e(\delta_{c1} t^e_{d2} - \delta_{c2} t^e_{d1}),\end{aligned}$$

$$\alpha^f t^f_{c1} = \alpha^1 t^1_{c1} + \alpha^2 t^2_{c1} + \alpha^3 t^3_{c1} = \tilde{\alpha}^1 \delta_{c2} + i\tilde{\alpha}^2 \delta_{c2} + \tilde{\alpha}^3 \delta_{c1},$$

$$\alpha^f t^f_{c2} = \alpha^1 t^1_{c2} + \alpha^2 t^2_{c2} + \alpha^3 t^3_{c2} = \tilde{\alpha}^1 \delta_{c1} - i\tilde{\alpha}^2 \delta_{c1} - \tilde{\alpha}^3 \delta_{c2}.$$

For the last two expressions we used $t^a = \sigma^a/2$ and $\tilde{\alpha}^a \coloneqq \alpha^a/2$. We also used how the Pauli matrices explicitly look like to find, for example, $\sigma^1_{c2} = \delta_{c1}$. Using these identities, we finally find

$$\begin{aligned}&\epsilon^{ab} \bar{q}_L'^c U_{q_L}^{\dagger ca} \phi^{\dagger d} U_\phi^{\dagger db}\\ &= \bar{q}_L'^c \phi^{\dagger d}\left((\delta_{d2}\delta_{c1} - \delta_{d1}\delta_{c2}) - i\alpha^f\left(\delta_{d2} t^f_{c1} - \delta_{d1} t^f_{c2}\right) - i\alpha^e(\delta_{c1} t^e_{d2} - \delta_{c2} t^e_{d1})\right)\\ &= \bar{q}_L'^1 \phi^{\dagger 2} - \bar{q}_L'^2 \phi^{\dagger 1} - i\alpha^f\left(\phi^{\dagger 2} \bar{q}_L'^c t^f_{c1} - \phi^{\dagger 1} \bar{q}_L'^c t^f_{c2}\right) - i\alpha^e\left(\bar{q}_L'^1 \phi^{\dagger d} t^e_{d2} - \bar{q}_L'^2 \phi^{\dagger d} t^e_{d1}\right)\\ &= \epsilon^{ab} \bar{q}_L'^a \phi^{\dagger b} - i\left(\phi^{\dagger 2}(\tilde{\alpha}^1 \bar{q}_L'^2 + i\tilde{\alpha}^2 \bar{q}_L'^2 + \tilde{\alpha}^3 \bar{q}_L'^1) - \phi^{\dagger 1}(\tilde{\alpha}^1 \bar{q}_L'^1 - i\tilde{\alpha}^2 \bar{q}_L'^1 - \tilde{\alpha}^3 \bar{q}_L'^2)\right)\\ &\qquad - i\left(\bar{q}_L'^1\left(\tilde{\alpha}^1 \phi^{\dagger 1} - i\tilde{\alpha}^2 \phi^{\dagger 1} - \tilde{\alpha}^3 \phi^{\dagger 2}\right) - \bar{q}_L'^2\left(\tilde{\alpha}^1 \phi^{\dagger 2} + i\tilde{\alpha}^2 \phi^{\dagger 2} + \tilde{\alpha}^3 \phi^{\dagger 1}\right)\right)\\ &= \epsilon^{ab} \bar{q}_L'^a \phi^{\dagger b} - i\big(\tilde{\alpha}^1\left(\bar{q}_L'^2 \phi^{\dagger 2} - \bar{q}_L'^1 \phi^{\dagger 1} + \bar{q}_L'^1 \phi^{\dagger 1} - \bar{q}_L'^2 \phi^{\dagger 2}\right)\\ &\qquad + i\tilde{\alpha}^2\left(\bar{q}_L'^2 \phi^{\dagger 2} + \bar{q}_L'^1 \phi^{\dagger 1} - \bar{q}_L'^1 \phi^{\dagger 1} - \bar{q}_L'^2 \phi^{\dagger 2}\right)\\ &\qquad + \tilde{\alpha}^3\left(\bar{q}_L'^1 \phi^{\dagger 2} + \bar{q}_L'^2 \phi^{\dagger 1} - \bar{q}_L'^1 \phi^{\dagger 2} - \bar{q}_L'^2 \phi^{\dagger 1}\right)\big)\\ &= \epsilon^{ab} \bar{q}_L'^a \phi^{\dagger b}.\end{aligned}$$

Thus, also the third term is indeed gauge invariant.

20 Quantization of GWS Theory

20.1 R-Xi Gauge – Faddeev-Popov Procedure

20.1.1 Functional Quantization

For quantization, we start with the functional integral

$$Z = \int \mathcal{D}A\,\mathcal{D}\phi' \exp\Big(i\int d^4x\,\mathcal{L}[A,\phi']\Big),$$

where A summarizes all fields A_μ^a and χ is the scalar field deviation from its vacuum expectation value: $\phi = \phi_0 + \phi'$. As in (>15.3.2) and (>18.2.1) we fix the gauge by introducing a 1 in form of

$$1 = \int \mathcal{D}\alpha\,\delta\big(G^a(A^\alpha,\phi'^\alpha)\big)\left|\det\frac{\delta G^c(A^\alpha,\phi'^\alpha)}{\delta\alpha^d}\right|.$$

We can now turn $A^\alpha \to A$ by a shift and rotation and $\phi'^\alpha \to \phi'$ by a unitary transformation, without changing the integration measure or Lagrangian or determinant, as explained in (>18.2.1):

$$\begin{aligned} Z &= \int \mathcal{D}A\,\mathcal{D}\phi' \int \mathcal{D}\alpha\,\delta\big(G^a(A^\alpha,\phi'^\alpha)\big)\exp\Big(i\int d^4x\,\mathcal{L}[A,\phi']\Big)\left|\det\frac{\delta G^c}{\delta\alpha^d}\right| \\ &= \int \mathcal{D}A\,\mathcal{D}\phi'\,\delta\big(G^a(A,\phi')\big)\exp\Big(i\int d^4x\,\mathcal{L}[A,\phi']\Big)\left|\det\frac{\delta G^c}{\delta\alpha^d}\right| \end{aligned}$$

We also dropped the integration over α here, since it drops out for n-point functions. Let us use the so-called R_ξ gauge condition,

$$G^c(A^\alpha,\phi'^\alpha) = H^c\big(A^\alpha(x),\phi'^\alpha(x)\big) - \omega^c(x), \qquad H^c(A^\alpha,\phi'^\alpha) := \frac{1}{\sqrt{\xi}}\big(\partial_\mu A_\alpha^{c\mu} + \xi g F_i^c \phi_i'^\alpha\big),$$

where $F_i^a := T_{ij}^a\phi_{0j}$ such that $m_{ab}^2 = g^2F^aF^b$, as we know from section 19.3.

Following the steps from (>18.2.1), we can integrate over ω with a Gaussian weight, such that (neglecting global normalization factors)

$$\begin{aligned} Z &= \int \mathcal{D}\omega \exp\Big(-i\int d^4x\frac{1}{2}\omega^b\omega_b\Big)\int \mathcal{D}A\mathcal{D}\phi'\delta(H^a(A,\phi')-\omega^a)\exp\Big(i\int d^4x\mathcal{L}[A,\phi']\Big)\left|\det\frac{\delta G^c}{\delta\alpha^d}\right| \\ &= \int \mathcal{D}A\,\mathcal{D}\phi'\,\exp\Big(-i\int d^4x\frac{1}{2}H^bH_b\Big)\exp\Big(i\int d^4x\,\mathcal{L}[A,\phi']\Big)\left|\det\frac{\delta G^c}{\delta\alpha^d}\right|. \end{aligned}$$

The gauge fixing term effectively adds the following terms to the Lagrangian:

$$\begin{aligned} -\frac{1}{2}H^aH_a &= -\frac{1}{2\xi}\big(\partial_\mu A^{a\mu} + \xi g F_i^a\phi_i'\big)^2 \\ &= -\frac{1}{2\xi}\Big(\big(\partial_\mu A^{a\mu}\big)\big(\partial_\nu A_a^\nu\big) + 2\xi g\big(F_{ia}\phi_i'\big)\big(\partial_\mu A^{a\mu}\big) + \xi^2g^2\big(F_i^a\phi_i'\big)\big(F_{ja}\phi_j'\big)\Big) \\ &= \frac{1}{2\xi}\big(A_\mu^a\partial^\mu\partial_\nu A_a^\nu\big) - g\big(F_{ia}\phi_i'\big)\big(\partial_\mu A^{a\mu}\big) - \frac{1}{2}\xi g^2\big(F_i^a\phi_i'\big)^2. \end{aligned}$$

20.1.2 Gauge Boson and Goldstone Boson Kinetic Terms

Those terms are added to the Lagrangian. The Lagrangian that is invariant under the gauge transformations that turned $A^\alpha \to A$ and $\phi'^\alpha \to \phi'$ reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^aF_a^{\mu\nu} + \frac{1}{2}\big(D_\mu\phi\big)^2 - V(\phi).$$

In the form in which we have given the Lagrangian above, it is made up of the following three terms

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) + \mathcal{O}(A^3)$$
$$= -\frac{1}{2}\Big(\left(\partial_\mu A^a_\nu\right)\left(\partial^\mu A^\nu_a\right) - \left(\partial_\mu A^a_\nu\right)\left(\partial^\nu A^\mu_a\right)\Big) + \mathcal{O}(A^3) = \frac{1}{2}A^a_\mu(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)A^a_\nu + \mathcal{O}(A^3),$$

$$\frac{1}{2}\left(D_\mu\phi\right)^2 = \frac{1}{2}\left(\partial_\mu\phi'\right)^2 - gA^\mu_a\left(\partial_\mu\phi'_i\right)F^a_i + \frac{m^2_{ab}}{2}A^a_\mu A^\mu_b + \mathcal{O}((\text{fields})^3),$$

$$V(\phi) = V(\phi_0 + \phi') = V(\phi_0) + \underbrace{\frac{1}{2}\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}V(\phi)\bigg|_{\phi=\phi_0}}_{=:M^2_{ij}}\underbrace{(\phi_i - \phi_{i0})}_{=\phi'_i}\underbrace{(\phi_j - \phi_{j0})}_{=\phi'_j} + \mathcal{O}(\phi'^3).$$

For the first one, we used the definition of $F^a_{\mu\nu}$ from section 3.7. For the second one we copied the result from section 18.3 or (>19.3.2). Finally, the last term was copied from (>19.2.1). Thus, was appears in the exponent of the path integral Z reads (neglecting the constant $V(\phi_0)$)

$$\mathcal{L} - \frac{1}{2}H^aH_a = \left(\frac{1}{2}A^a_\mu(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)A^a_\nu + \frac{1}{2}\left(\partial_\mu\phi'\right)^2 - gA^\mu_a\left(\partial_\mu\phi'_i\right)F^a_i + \frac{m^2_{ab}}{2}A^a_\mu A^\mu_b - \frac{1}{2}M^2_{ij}\phi'_i\phi'_j\right)$$
$$+\left(\frac{1}{2\xi}\left(A^a_\mu\partial^\mu\partial_\nu A^\nu_a\right) - g(F_{ia}\phi'_i)\left(\partial_\mu A^{a\mu}\right) - \frac{1}{2}\xi g^2(F^a_i\phi'_i)^2\right) + \mathcal{O}((\text{fields})^3)$$
$$= \frac{1}{2}A^a_\mu(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)\delta^{ab}A^b_\nu + \frac{1}{2}\left(\partial_\mu\phi'\right)^2 + \frac{m^2_{ab}}{2}\eta^{\mu\nu}A^a_\mu A^b_\nu - \frac{1}{2}M^2_{ij}\phi'_i\phi'_j$$
$$+\frac{1}{2\xi}\left(A^a_\mu\partial^\mu\partial^\nu A^b_\nu\right)\delta^{ab} - \frac{1}{2}\xi g^2(F^a_i\phi'_i)^2 + \mathcal{O}((\text{fields})^3)$$
$$= -\frac{1}{2}A^a_\mu\left(\left(-\eta^{\mu\nu}\Box + \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)\delta^{ab} - m^2_{ab}\eta^{\mu\nu}\right)A^b_\nu$$
$$+\frac{1}{2}\left(\partial_\mu\phi'\right)^2 - \frac{1}{2}M^2_{ij}\phi'_i\phi'_j - \frac{1}{2}\xi g^2(F^a_i\phi'_i)^2 + \mathcal{O}((\text{fields})^3).$$

The terms that mix A^μ_a and ϕ'_i – one of which comes from the original Lagrangian and one out of H^aH_a – cancel by partial integration.

20.1.3 Ghost Kinetic Terms

Finally, we must take care of the functional determinant. When we wrote the fields $A^{a\mu}_\alpha, \phi'^\alpha$ with an additional index α, we meant by them the gauge transformed fields, that depend on the field $\alpha^a(x)$. Specifically, for the gauge fields it means (exactly as in (>18.2.1))

$$A^{a\mu}_\alpha = A^{a\mu} - \frac{1}{g}\left(D_\mu\alpha\right)^a \qquad \Rightarrow \qquad \delta A^{a\mu}_\alpha = -\frac{1}{g}\left(D_\mu\alpha\right)^a.$$

Note, that this form of the gauge transformation of $A^{a\mu}_\alpha$ was derived in (>18.2.1) and holds, if the generator t^a inside the covariant derivative D_μ is chosen in the adjoint representation.

ϕ'^a can be read of the gauge transformation rule for ϕ (from (>19.3.1)) since $\phi = \phi_0 + \chi$:

$$\phi \to \phi - \alpha^aT^a\phi \qquad \Leftrightarrow \qquad \phi_0 + \phi' \to \phi_0 + \phi' - \alpha^aT^a(\phi_0 + \phi')$$
$$\Leftrightarrow \qquad \phi' \to \phi' - \alpha^aT^a(\phi_0 + \phi') \qquad \Rightarrow \qquad \delta\phi'^\alpha = -\alpha^aT^a(\phi_0 + \phi').$$

Thus,

$$\frac{\delta G^c}{\delta\alpha^d} = \frac{\delta}{\delta\alpha^d}\left(\frac{1}{\sqrt{\xi}}\left(\partial_\mu A^{c\mu}_\alpha + \xi gF^c_i\phi'^\alpha_i\right) - \omega^c\right) = \frac{1}{\sqrt{\xi}}\left(-\frac{1}{g}\partial_\mu D^\mu_{cd} - \xi gF^c_iT^d_{ij}(\phi_0 + \phi')_j\right).$$

Using, as already in (>18.2.2) and (>15.5.2), the analogy $\int(\Pi_i d\theta^*_i d\theta_i)e^{-\theta^*_iA_{ij}\theta_j} = \det A$, we can write the determinant as a functional integral

$$\left|\det\frac{\delta G^c}{\delta\alpha^d}\right| = \left|\det i\frac{\delta G^c}{\delta\alpha^d}\right| = \int \mathcal{D}\vartheta\,\mathcal{D}\bar{\vartheta}\,\exp\left(i\int d^4x\,\bar{\vartheta}\big(-\partial_\mu D^\mu_{cd} - \xi g^2 F_i^c T_{ij}^d(\phi_0+\phi')_j\big)\vartheta\right).$$

Here, we absorbed a factor of $1/g\sqrt{\xi}$ into the fields ϑ.[1]

20.2 R-Xi Gauge – Propagators

20.2.1 Gauge Boson Propagator

The terms of order A^2 in the Lagrangian where computed at the end of (>20.1.2) read

$$-\frac{1}{2}A_\mu^a\left(\left(-\eta^{\mu\nu}\Box + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)\delta^{ab} - m_{ab}^2\eta^{\mu\nu}\right)A_\nu^b.$$

In Fourier space, the bracket becomes

$$\begin{aligned}&\left(\eta^{\mu\nu}k^2 - \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right)\delta^{ab} - m_{ab}^2\eta^{\mu\nu}\\ &\qquad = \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)(k^2\delta^{ab} - m_{ab}^2) + \frac{k^\mu k^\nu}{k^2}\frac{1}{\xi}(k^2\delta^{ab} - \xi m_{ab}^2).\end{aligned}$$

Let us write the expression above as a matrix, that is $\delta^{ab}\to 1$ and $m_{ab}^2 \to \widetilde{m}_A^2$.

We are looking for the Greens function $i\widehat{D}_F^{\mu\nu}$ with matrix components $i\widehat{D}_{F,ab}^{\mu\nu}$ of this operator, just as at the very end of (>18.2.1). The following expression does the job:

$$\widehat{D}_F^{\mu\nu} = \frac{-i}{k^2-\widetilde{m}_A^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \frac{-i\xi}{k^2 - \xi\widetilde{m}_A^2}\frac{k^\mu k^\nu}{k^2},$$

as we can show readily explicitly:

$$\begin{aligned}&\left(\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)(k^2-\widetilde{m}_A^2) + \frac{k^\mu k^\nu}{k^2}\frac{1}{\xi}(k^2-\xi\widetilde{m}_A^2)\right)\overbrace{i\left(\frac{-i}{k^2-\widetilde{m}_A^2}\left(\eta_{\nu\sigma} - \frac{k_\nu k_\sigma}{k^2}\right) + \frac{-i\xi}{k^2-\xi\widetilde{m}_A^2}\frac{k_\nu k_\sigma}{k^2}\right)}^{=iD_{F,\nu\sigma}}\\ &\quad = \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\left(\eta_{\nu\sigma} - \frac{k_\nu k_\sigma}{k^2}\right) + \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\frac{(k^2-\widetilde{m}_A^2)\xi}{k^2-\xi\widetilde{m}_A^2}\frac{k_\nu k_\sigma}{k^2}\\ &\qquad + \frac{k^\mu k^\nu}{k^2}\frac{1}{\xi}\frac{k^2-\xi\widetilde{m}_A^2}{k^2-\widetilde{m}_A^2}\left(\eta_{\nu\sigma} - \frac{k_\nu k_\sigma}{k^2}\right) + \frac{k^\mu k^\nu}{k^2}\frac{k_\nu k_\sigma}{k^2}\\ &\quad = \left(\delta_\sigma^\mu - \frac{k^\mu k_\sigma}{k^2} - \frac{k^\mu k_\sigma}{k^2} + \frac{k^\mu k_\sigma}{k^2}\right) + \left(\frac{k^\mu k_\sigma}{k^2} - \frac{k^\mu k_\sigma}{k^2}\right)\frac{(k^2-\widetilde{m}_A^2)\xi}{k^2-\xi\widetilde{m}_A^2}\\ &\qquad + \frac{1}{\xi}\frac{k^2-\xi\widetilde{m}_A^2}{k^2-\widetilde{m}_A^2}\left(\frac{k^\mu k_\sigma}{k^2} - \frac{k^\mu k_\sigma}{k^2}\right) + \frac{k^\mu k_\sigma}{k^2} = \delta_\sigma^\mu.\end{aligned}$$

The form of $\widehat{D}_F^{\mu\nu}$ given above was such that it was easy to show that it is a Greens function of the respective operator. However, it possible to give it in a somewhat nicer form:

[1] *To me, it comes as a little surprise, that we also can absorb the gauge ξ into the ghost fields. Peskin&Schröder let this factor $1/\sqrt{\xi}$ simply disappear, and the explanation, that it is absorbed into the fields, is the best I have.*

$$\begin{aligned}
\widehat{D}_F^{\mu\nu} &= \frac{-i}{k^2-\widetilde{m}_A^2}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)+\frac{-i\xi}{k^2-\xi\widetilde{m}_A^2}\frac{k^\mu k^\nu}{k^2} \\
&= \frac{-i}{k^2-\widetilde{m}_A^2}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}+\xi\frac{k^2-\widetilde{m}_A^2}{k^2-\xi\widetilde{m}_A^2}\frac{k^\mu k^\nu}{k^2}\right) \\
&= \frac{-i}{k^2-\widetilde{m}_A^2}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\frac{1}{k^2-\xi\widetilde{m}_A^2}\left((k^2-\xi\widetilde{m}_A^2)-\xi(k^2-\widetilde{m}_A^2)\right)\right) \\
&= \frac{-i}{k^2-\widetilde{m}_A^2}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2-\xi\widetilde{m}_A^2}(1-\xi)\right).
\end{aligned}$$

This propagator is, just as the mass matrix $\widetilde{m}_A^2$, a matrix with indices a, b; that is, the components of $\widehat{D}_F^{\mu\nu}$ are $\widehat{D}_{F,ab}^{\mu\nu}$:

$$\widehat{D}_{F,ab}^{\mu\nu} = \left(\frac{-i}{k^2-\widetilde{m}_A^2}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2-\xi\widetilde{m}_A^2}(1-\xi)\right)\right)^{ab}.$$

20.2.2 Scalar Field Propagator

The effective Lagrangian $\mathcal{L}_{\text{eff}} = \mathcal{L} - H_a H^a/2 +$ (ghost terms) contains the following terms containing only two scalar fields ϕ' (we can read them of from section 19.1 or (>20.1.2)):

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \frac{1}{2}\left(\partial_\mu\phi'\right)^2 - \frac{1}{2}\xi g^2 (F_i^a\phi_i')^2 - \frac{1}{2}M_{ij}^2\phi_i'\phi_j' + \cdots \\
&= \frac{1}{2}\left(\partial_\mu\phi'\right)^2 - \frac{1}{2}\left(\xi g^2 F_i^a F_j^a + \frac{1}{2}M_{ij}^2\right)\phi_i'\phi_j' + \cdots.
\end{aligned}$$

By analogy to the Lagrangians and propagators of the massive Klein-Gordon field, the propagator to this Lagrangian reads

$$D_F^{ij} = \left(\frac{i}{k^2-\xi g^2 F^a F^a - M^2}\right)^{ij}.$$

20.2.3 Ghost Propagator

The terms of the ghost part of the effective Lagrangian from section 19.1 or (>20.1.3) that contain exactly two ghost fields (and no other fields) read

$$\bar{\vartheta}\left(-\partial_\mu\partial^\mu - \xi g^2 F_i^a T_{ij}^b \phi_{0j}\right)\vartheta = \left(\partial_\mu\bar{\vartheta}\right)\left(\partial^\mu\vartheta\right) - \xi g^2 F_i^a F_i^b\, \bar{\vartheta}\vartheta,$$

where we used partial integration for the first term. It propagator reads

$$\breve{D}_{F,ab} = \left(\frac{i}{k^2-\xi g^2 F_i^a F_i^b}\right)^{ab}.$$

20.3 R-Xi Gauge – Propagators for GWS Theory

20.3.1 Propagators for GWS Theory

We know from section 18.4 that the mass matrix of the gauge bosons of GWS theory reads

$$m_{ab}^2 = (\widetilde{m}_A^2)^{ab} = \frac{v^2}{4}\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix}^{ab},$$

where $a = 1, 2, 3, 4$ with $A_\mu^4 := B_\mu$. As in (>19.4.4), we can diagonalize this matrix and thereby find the mass eigenstates, which we called $W_\mu^\pm, Z_\mu^0$ and A_μ, with the masses m_A, m_Z, m_W given in section 18.4. Then, after diagonalization, also the propagator $\widehat{D}_{F,ab}^{\mu\nu}$ becomes diagonal. We then can write the gauge boson propagators as

$$\widehat{D}_F^{\mu\nu} = \frac{-i}{k^2 - m_X^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi m_X^2}(1 - \xi)\right),$$

where $m_X^2 = m_W^2, m_Z^2, m_A^2$ for the $W^\pm$ bosons, the Z boson or the photon respectively ($m_A = 0$). The mass matrix $m_{ab}^2 = g^2 F_i^a F_i^b$ also appears in the ghost propagator, which we can write after diagonalization as

$$\breve{D}_F = \frac{i}{k^2 - \xi m_X^2}.$$

Finally, the mass of the propagator of the scalar field reads

$$-\xi g^2 F_i^a F_j^a - M_{ij}^2.$$

The g in this term equals the g^a with $g^a = g$ for $a = 1, 2, 3$ and $g^4 = g'$ which we used in chapter 19. That is, using the matrix $g^a F_i^a$ from section 18.4,

$$\begin{aligned}
g^2 F_i^a F_j^a &:= \sum_a (g^a F_i^a)(g^a F_j^a) = (gF)^{ai}(gF)^{aj} = \left((gF)^T(gF)\right)^{ij} \\
&= \frac{v^2}{4}\left(\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & g & -g' \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & -g' & 0 \end{pmatrix}\right)^{ij} = \frac{v^2}{4}\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 + g'^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{ij} \\
&= \begin{pmatrix} m_W^2 & 0 & 0 & 0 \\ 0 & m_W^2 & 0 & 0 \\ 0 & 0 & m_Z^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{ij}.
\end{aligned}$$

Recall our construction of the scalar field from (>19.3.1),

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} -i\phi_1 - \phi_2 \\ \phi_4 + i\phi_3 \end{pmatrix},$$

with four real fields ϕ_i. We always chose the vacuum expectation value to only have the component ϕ_4 non-vanishing. Thus, in analogy with our derivation in section 18.1, the components ϕ_1, ϕ_2, ϕ_3 describe Goldstone bosons, which do not obtain a mass M_{ij} from the potential V (>19.2.1). However, they do obtain a mass $g^2 F_i^a F_j^a$ from the Higgs mechanism.

On the other hand, the Higgs field is described by the fourth component ϕ_4. Obviously, the fourth diagonal component of the matrix $g^2 F_i^a F_j^a$ is zero, but the Higgs boson receives a mass from M_{ij}.

Thus, the Goldstone boson propagator reads

$$D_F = \frac{i}{k^2 - \xi m_G^2}, \qquad m_G = m_W, m_W, m_Z$$

and the propagator of the Higgs boson reads

$$D_F = \frac{i}{k^2 - m_h}$$

with an independent Higgs mass m_h.

21

21.2 ???

21.2.1 ???

Consider an infinitesimal SU(N) gauge transformation. Whereas the transformation of the gauge fields is more complex under this transformation (see section 3.9 or later below), scalar or fermion fields transform under such a transformation like

$$\varphi_i(x) \to \varphi_i(x) + \delta\varphi_i(x), \qquad \text{where} \qquad \delta\varphi_i(x) = -i\alpha_a(x)\, t^a_{ij}\, \varphi_j(x).$$

If $\alpha_a(x)$ is a *constant* function of x, that is $\alpha_a(x) \equiv \alpha_a$, the transformation is *global*.

Consider a Lagrangian $\mathcal{L}_0$ that is symmetric under such a *global* transformation. Usually, that means that $\mathcal{L}_0$ simply contains the kinetic and the mass terms of the scalar or/and fermion fields φ_i, since they are quadratic in those fields and therefore respect this symmetry.

According to section 3.3 (in the present case for $x'^\mu = x^\mu \Longrightarrow \delta x^\mu = 0$), this yields a variation of the Lagrangian $\mathcal{L}_0 \to \mathcal{L}_0 + \delta\mathcal{L}_0$ with

$$\delta\mathcal{L}_0 = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_i\right)}\delta\varphi_i(x)\right) = -i\alpha_a\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_i\right)}t^a_{ij}\varphi_i(x)\right).$$

Since α_a is arbitrary and the unitary transformation should leave the Lagrangian unchanged (as we assumed in the beginning), we can identify the current

$$J^{a\mu}(x) \coloneqq -i\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_i\right)}t^a_{ij}\varphi_i(x) \qquad \text{with} \qquad \partial_\mu J^{a\mu}(x) = 0.$$

Using $\mathcal{L}_0$ as a starting point, we now want to construct a general Lagrangian $\mathcal{L}$ that is also symmetric under a *local* gauge transformation, where $\alpha_a(x)$ *is* a function of x. Our Lagrangian $\mathcal{L}_0$, that is symmetric under a *global* gauge transformation, now receives an additional term $\delta\mathcal{L}_0 \neq 0$:

$$\begin{aligned}\delta\mathcal{L}_0 &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_i\right)}\delta\varphi_i(x)\right) = \partial_\mu\underbrace{\left(-i\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_i\right)}\alpha_a(x)t^a_{ij}\varphi_i(x)\right)}_{=\alpha_a J^{a\mu}(x)} \\ &= \alpha_a(x)\,\underbrace{\partial_\mu J^{a\mu}(x)}_{=0} + J^{a\mu}(x)\,\partial_\mu\alpha_a(x) = J^{a\mu}(x)\,\partial_\mu\alpha_a(x).\end{aligned}$$

Thus, we need to add another term $\mathcal{L}'$ to construct $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$, such that $\mathcal{L}$ is invariant under local gauge transformation. For this term, we should use a gauge field, that transforms as

$$A^a_\mu \to A^a_\mu - \frac{1}{g}\left(\partial_\mu\alpha^a\right) + f^{abc}A^b_\mu\alpha^c$$

under local gauge transformation, as we found in section 3.9. If we use $\mathcal{L}' = gA^a_\mu J^{a\mu} + \mathcal{O}(A^2)$, we find that

$$\mathcal{L}' \to\ g\left(A^a_\mu - \frac{1}{g}\left(\partial_\mu\alpha^a\right) + f^{abc}A^b_\mu\alpha^c\right)J^{a\mu}$$

$$\delta\mathcal{L} = \delta\mathcal{L}_0 + \delta\mathcal{L}' = J^{a\mu}\,\partial_\mu\alpha_a +\ g\left(-\frac{1}{g}\left(\partial_\mu\alpha^a\right) + f^{abc}A^b_\mu\alpha^c\right)J^{a\mu} = gf^{abc}A^b_\mu\alpha^c\,J^{a\mu}$$